Representation Theory of Linear Algebraic Groups : I

Lecture 1:

Organisation:

- 2 lectures / week
- Oral exam (date TBD)
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Overview: "Reductive" Symmetry:

{reductive linear algebraic groups} $\ni GL_n, SL_n, O_n, Sp_n, \ldots$

variety + group structure defined by polynomials

$\Downarrow$

{real reductive groups} $\ni O(3,1), GL_n(\mathbb{R})^+, \ldots$

$\cup$

{compact Lie groups} $\ni O(n), U(n), \ldots$

compact manifold + continuous group structure

{finite groups of Lie type} $\ni PSL_n(\mathbb{F}_q), \ldots$

almost all finite simple groups

{reductive Lie algebras} $\ni \mathfrak{gl}_n, \mathfrak{so}_n, \mathfrak{sp}_n, \ldots$

Prototype example.

$$\mathcal{G} = SL_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \det = ad - bc = 1 \right\}$$

$\Big\downarrow$ Interesting subgroups

maximal torus $\quad T = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \right\} \cong \mathcal{G}_m$

$$U = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\} \cong \mathcal{G}_a$$

$$U^- = \left\{ \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix} \right\} \cong \mathcal{G}_a$$

Borel subgroup $\quad B = \left\{ \begin{pmatrix} a & b \\ & a^{-1} \end{pmatrix} \right\} \cong \mathcal{G}_m \ltimes \mathcal{G}_a$

$$N(T) = \left\{ \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}, \begin{pmatrix} & b \\ -b^{-1} & \end{pmatrix} \right\}$$

Weyl group $\quad W = N(T)/T \cong S_2$

$$\mathfrak{g} = \mathfrak{sl}_2 = \mathrm{Lie}(SL_2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a+d=0 \right\}$$

$\Big\}$ subalgebras $\downarrow$

Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(T) = \left\{ \begin{pmatrix} a & \\ & -a \end{pmatrix} \right\}$

$$\mathfrak{u} = \mathrm{Lie}(U) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}$$

$$\mathfrak{u}^- = \mathrm{Lie}(U) = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right\}$$

Borel subalgebra $\mathfrak{b} = \mathrm{Lie}(B) = \left\{ \begin{pmatrix} a & b \\ & -a \end{pmatrix} \right\}$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

e: "raising operator" $\quad$ f: "lowering operator"

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

General case:

$$G \text{ reductive group}$$

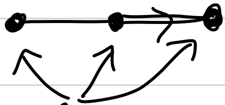
$$\mathfrak{g} = \mathrm{Lie}\, G \text{ reductive Lie algebra}$$

G is generated by T and copies of PGL_2 / SL_2.

$\mathfrak{g}$ —— " —— $\mathfrak{h}$ —— " —— $\mathfrak{sl}_2$.

- Copies of SL_2 correspond to roots / coroots

  in root datum / root system / Dynkin diagram



$\mathfrak{sl}_2$

- "Angle" between roots $\Leftrightarrow$ Interaction of copies

- $\mathfrak{sl}_\alpha \rightsquigarrow \langle \mathfrak{g}_\beta \rangle_{\mathfrak{sl}_\alpha}$ describes interaction.

$\uparrow$

f.d. $\mathfrak{sl}_2$-rep.

Lecture 2

<u>Example/Tensor</u>: $SL_3 = \{3\times 3 \text{ matrix}, \det = 1\}$

$$g_\alpha = \left\{ \begin{pmatrix} \multicolumn{2}{c|}{*} & 0 \\ & & 0 \\ \hline 0 & 0 & 1 \end{pmatrix} \right\} \cong SL_2$$

$$g_\beta = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \hline 0 & \multicolumn{2}{|c}{*} \\ 0 & & \end{pmatrix} \right\} \cong SL_2$$

$\Big\downarrow$ Lie

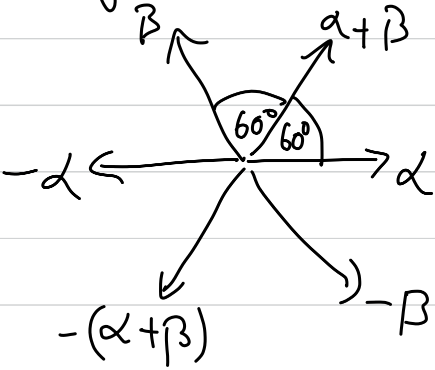
$$\mathfrak{sl}_\alpha \longleftrightarrow \langle g_\beta \rangle_{\mathfrak{sl}_\alpha} = \left\{ \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

$$\mathfrak{sl}_\alpha = \left\{ \begin{pmatrix} \multicolumn{2}{c|}{*} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{pmatrix} \right\} \qquad g_\beta = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

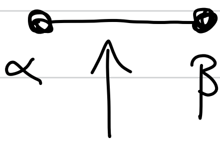
$\uparrow$ 2-dim $\mathfrak{sl}_2$-rep.

?

root system



Dynkin diagram



$\#\text{edges} = \dim \langle \mathfrak{g}_\beta \rangle_{\mathfrak{sl}_\alpha} - 1$

And much more!

1. Lie Algebras

Global assumption: Everything $/\mathbb{C}$

1.1. Basic definitions

Def A: A Lie algebra $\mathfrak{g}$ is a $\mathbb{C}$-v.s. with an operation, called (Lie-)bracket,

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

such that $[\ ,\ ]$

(1) is bilinear, so

$$[aX+Y, Z] = a[X,Z] + [Y,Z], \quad [Z, aX+Y] = \dots$$

(2) fulfills the Jacobi identity

$$[X,[Y,Z]] = [[X,Y],Z] + [Y,[X,Z]]$$

(3) is alternating, so

$$[X, X] = 0$$

Lemma: Let A be an associative algebra. Then A equipped with $[\,,\,]$ is a Lie algebra, where

$$[X, Y] = XY - YX.$$

Pf: (1) Bilinear: ✓

(2) Jacobi: $[[X,Y],Z] + [Y,[X,Z]]$

$$= (XY - YX)Z - Z(XY - YX) + Y(XZ - ZX) - (XZ - ZX)Y$$

$$= XYZ - YXZ - ZXY + ZYX + YXZ - YZX - XZY + ZXY$$

$$= X(YZ - ZY) - (YZ - ZY)X = [X, [Y, Z]]$$

(3) Alternating ✓ $\square$

<u>Example</u>: (1) Let V be a vector space.

Write $\mathfrak{gl}(V)$ for the Lie algebra associated to $\operatorname{End}(V)$. Similarly, write $\mathfrak{gl}_n(\mathbb{C})$ for $\mathbb{C}^{n\times n}$.

(2) Let G be a Lie group, then T_eG can be equipped with the structure of a Lie algebra.

(3) The 3-dim. Lie algebra $\mathfrak{g}$, spanned by u, v, z with bracket defined by

$$[u,v] = z, \quad [u,z] = 0, \quad [v,z] = 0$$

is called the <u>Heisenberg algebra</u>.

Def B Let $\mathfrak{u} \subset \mathfrak{g}$ be a sub vector space of a Lie algebra.

(1) $\mathfrak{u}$ is called a (Lie-) subalgebra if

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u}$$

(2) $\mathfrak{u}$ is called an ideal in $\mathfrak{g}$ if

$$[\mathfrak{g}, \mathfrak{u}] \subset \mathfrak{u}$$

Lemma B: If $\mathfrak{u} \subset \mathfrak{g}$ is an ideal, then the quotient vector space $\mathfrak{g}/\mathfrak{u}$ inherits the structure of a Lie algebra from $\mathfrak{g}$.

ideals in $\mathfrak{g}$ $\rightsquigarrow$ normal subgroups in G

Example B: (1) Denote

$$\mathfrak{sl}_n(\mathbb{C}) = \{X \mid \operatorname{tr} X = 0\} \subset \mathfrak{gl}_n(\mathbb{C})$$

$$\mathfrak{so}_n(\mathbb{C}) = \{X \mid X = -X^{tr}\} \subset \mathfrak{gl}_n(\mathbb{C})$$

$$\mathfrak{sp}_{2n}(\mathbb{C}) = \{X \mid X J_{2n} = -J_{2n} X^{tr}\} \subset \mathfrak{gl}_n(\mathbb{C})$$

where $J_{2n} = \begin{pmatrix} & -I_n \\ I_n & \end{pmatrix}$. Then these are subalgebras of $\mathfrak{gl}_n(\mathbb{C})$

(2) $\mathfrak{n} = \left\{\begin{pmatrix} & b \\ & \end{pmatrix}\right\} \subset \mathfrak{b} = \left\{\begin{pmatrix} a & b \\ & -a \end{pmatrix}\right\}$ is an ideal. (Recall that $[H, E] = 2E$).

$\mathfrak{h} = \left\{\begin{pmatrix} a & \\ & -a \end{pmatrix}\right\} \subset \mathfrak{b}$ is a subalgebra but not an ideal.

Def C: (1) Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras.

A morphism/map of Lie algebras is a linear map

$$\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$$

preserving the Lie bracket, so

$$\varphi([X,Y]) = [\varphi(X), \varphi(Y)]$$

(2) A representation V of $\mathfrak{g}$ is a map (of Lie algebras), $\mathfrak{g} \to \mathfrak{gl}(V)$

Remark: Specifying a map of Lie algebras $\mathfrak{g} \to \mathfrak{gl}(V)$ is equivalent to specifying an action $\mathfrak{g} \times V \to V$, $(X, v) \mapsto Xv$

fulfilling the rule

$$X(Yv) - Y(Xv) = [X, Y]v .$$

We will use both notations often.

Example C: (1) Let $\mathfrak{g}$ be a Lie algebra. Denote $ad(X)Y = [X, Y]$. Then

$$\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad X \mapsto ad(X)$$

defines a representation called the adjoint representation. Exercise: Prove this!

Moreover, the center of $\mathfrak{g}$ is the kernel of ad:

$$\mathfrak{z}(\mathfrak{g}) = \{ X \in \mathfrak{g} \mid [X, Y] = 0 \ \forall Y \} = \ker(ad)$$

(2) For $sl_2(\mathbb{C})$ we obtain (with the basis e, h, f)

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad \mathrm{ad}(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} \quad \mathrm{ad}(f) = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & 2 & 0 \end{pmatrix}$$

(3) Let $\mathfrak{g} = \langle u, v, z \rangle$ be the Heisenberg algebra. Then we obtain a faithful representation:

$$u \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad v \mapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad z \mapsto \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(4) For the Heisenberg algebra, we obtain the <u>Schrödinger representation</u> on $\mathbb{C}[x]$

$$u \mapsto \frac{\partial}{\partial x} \qquad v \mapsto x, \quad z \mapsto \mathrm{id}$$

$\square$

Lecture 3

0.2 Universal enveloping algebra

<u>Reminder:</u> Let V be a vector space. Then we can consider the n-th tensor power

$$T^n V = V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ copies of } V}$$

where by convention $T^0 V = \mathbb{C}$. We often omit the $\otimes$ from the notation and abbreviate

$$v_1 v_2 \cdots v_n = v_1 \otimes v_2 \cdots \otimes v_n \in T^n V.$$

The tensor algebra is defined as

$$T^\bullet V = \bigoplus_{n \geq 0} T^n V$$

with multiplication

$$(v_1 v_2 \cdots v_n)(v_1' v_2' \cdots v_m') = v_1 \cdots v_n v_1' \cdots v_m'.$$

$T^\bullet V$ is the free associative algebra generated by V.

Moreover, the <u>symmetric algebra</u> arises as quotient

$$\mathrm{Sym}^\bullet V = T^\bullet V \Big/ \langle xy - yx \mid x, y \in V \rangle$$

of the two-sided ideal generated by $xy - yx \in T^2 V$.

$\mathrm{Sym}^\bullet V$ is the free commutative algebra generated by V.

If $\{x_i\}_{i \in I}$ is a basis of V, then

$$\mathrm{Sym}^\bullet V \cong \mathbb{C}[x_i \mid i \in I].$$

Now let $\mathfrak{g}$ be a Lie algebra.

<u>Def</u>: The <u>universal enveloping algebra</u> $U(\mathfrak{g})$ of $\mathfrak{g}$ is the associative algebra defined by

$$T^\bullet \mathfrak{g} \Big/ \langle \underbrace{xy - yx}_{\in T^2\mathfrak{g}} = \underbrace{[x,y]}_{\in T^1\mathfrak{g}} \mid x, y \in \mathfrak{g} \rangle$$

$\square$

<u>Rem</u>: We omit the $\otimes$ in the notation for elements in $T^\bullet V$

<u>Prop</u>: $U(\mathfrak{g})$ is determined by the universal property

$$\begin{tikzcd}
\mathfrak{g} \arrow[r] \arrow[dr] & U(\mathfrak{g}) \arrow[d, dashed, "\exists !"] \\
& A
\end{tikzcd}$$

Lie alg. map   unital associative algebra

$\square$

Thm: (Poincaré - Birkhoff - Witt) Denote by $U_n(g) \subset U(g)$ the image of $\bigoplus_{i \leq n} T^i(g)$ under $T^\bullet g \twoheadrightarrow U(g)$. Then

(1) $U_n(g) \subset U_m(g)$ for $n \leq m$

(2) $U_n(g) U_m(g) \subset U_{n+m}(g)$ $\forall n, m$

(3) $\operatorname{gr} U(g) = \bigoplus_n U_n(g)/U_{n-1}(g)$ is a commutative graded algebra

(4) The natural map $\operatorname{Sym}(g) \to \operatorname{gr} U(g)$ is an isom. of graded algebras $\square$

The Thm. implies that as vector space, $U(g)$ and $\operatorname{Sym}(g)$ behave the same.

0.3 Automorphisms

Let $\dim \mathfrak{g} < \infty$

<u>Def</u> The group of (inner) automorphism of $\mathfrak{g}$ is

$$\begin{array}{ccccc} \mathrm{Int}(\mathfrak{g}) & \subset & \mathrm{Aut}(\mathfrak{g}) & \subset & GL(\mathfrak{g}) \\ \parallel & & \uparrow & & \\ \langle e^{\mathrm{ad}(x)} \mid x \in \mathfrak{g} \rangle & & \text{Lie algebra autom.} & & \end{array}$$

$\square$

<u>Prop</u>: $\mathrm{Int}(\mathfrak{g})$, $\mathrm{Aut}(\mathfrak{g})$ are Lie groups with

Lie algebras

$$\begin{array}{ccccc} \mathrm{Lie}(\mathrm{Int}(\mathfrak{g})) & \longrightarrow & \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})) & \rightarrow & \mathfrak{gl}(\mathfrak{g}) \\ \cong & & \cong & & \\ \mathfrak{g}/\mathfrak{z} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(\mathfrak{g}) & & \end{array}$$

where $\mathrm{Der}(\mathfrak{g}) = \{ \varphi \in \mathfrak{gl}(\mathfrak{g}) \mid \varphi([x,y]) = [\varphi(x), y] + [x, \varphi(y)] \}$

$\square$

Remark: (1) The exponential $e^X = \sum_{j=0}^{\infty} \frac{1}{j!} X^j$ for $X \in \mathrm{End}(V)$ works well when either

(a) $\dim V < \infty$

(b) X is locally nilpotent, that is $\forall v \in V \; \exists n \geq 0$, s.t. $X^n v = 0$

(2) Under this conditions $e^{\mathrm{ad}(X)} = \mathrm{Ad}(e^X)$ and similar standard formulas hold $\square$

Lecture 4

0.4. Representation theory of sl_2

We disuss the rep.-th. of

$$sl_2(\mathbb{C}) = \langle e = \begin{pmatrix} & 1 \\ & \end{pmatrix}, h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, f = \begin{pmatrix} & \\ 1 & \end{pmatrix} \rangle$$

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h$$

<u>Lemma</u>: The following relations hold in $U(sl_2(\mathbb{C}))$

(1) $[h, e^k] = 2k\, e^k$, $\qquad [h, f^k] = -2k\, f^k$

(2) $[e, f^k] = k\, f^{k-1}(h+1-k) = k\,(h+k-1)\, f^{k-1}$

$\qquad [f, e^k] = -k\, e^{k-1}(h+k-1) = -k\,(h+1-k)\, e^{k-1}$

<u>Proof</u>: (1) $[h, e^k] = [h,e]e^{k-1} + e\,[h, e^{k-1}]$

$$= \underset{\substack{\uparrow \\ \text{relation}}}{2e}\, e^{k-1} + e\, \underset{\substack{\uparrow \\ \text{induction}}}{2(k-1)\, e^{k-1}} = 2k\, e^k$$

(2) $[e, f^k] = [e,f] f^{k-1} + f \cdot [e, f^{k-1}]$

$$= h f^{k-1} + f (k-1)(h+k-2) f^{k-2}$$

$$\overset{fh = hf+2f}{=} h f^{k-1} + (k-1) h f^{k-1} + (k-1) 2 f^{k-1} + (k-1)(k-2) f^{k-1}$$

$$= k(h+k-1) f^{k-1}$$

The other equations are analogous $\square$

<u>Prop.</u> Let V be a $\mathfrak{sl}_2$-rep. with $v \in V$, $\lambda \in \mathbb{C}$ such that $hv = \lambda v$. Let $v_j = \frac{1}{j!} f^j v$.

Then

$$h v_j = (\lambda - 2j) v_j$$

If $ev = 0$, then we have

$$e v_j = (\lambda - j + 1)\, v_{j-1}$$

<u>Proof</u>: Use Lemma (1) and (2) $\square$

<u>Rem</u>: (1) If $hv = \lambda v$ and $ev = 0$, v is called a highest weight vector (of weight λ)

(2) One can visualise this as:

$$\begin{tikzcd}[column sep=large]
\cdots \arrow[r, phantom] & v_{j+1} \arrow[loop above, "\lambda-2(j+1)"] \arrow[r, bend left, "\lambda-j"] & v_j \arrow[loop above, "\lambda-2j"] \arrow[l, bend left, "j"] & \cdots \arrow[r, bend left, "\lambda-2"] & v_2 \arrow[loop above, "\lambda-4"] \arrow[l, bend left, "3"] \arrow[r, bend left, "\lambda-1"] & v_1 \arrow[loop above, "\lambda-2"] \arrow[l, bend left, "2"] \arrow[r, bend left, "\lambda"] & v_0 \arrow[loop above, "\lambda"] \arrow[l, bend left, "1"]
\end{tikzcd}$$

where $e: \longrightarrow$, $h: \circlearrowright$, $f: \longleftarrow$

Def A Let $\lambda \in \mathbb{C}$, $\mathfrak{b} = \langle h, e \rangle \subset \mathfrak{sl}_2$. Then

$$M(\lambda) = U(\mathfrak{sl}_2) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

is the Verma module (or universal highest weight module) for $\mathfrak{sl}_2$ of highest weight λ. Here $\mathfrak{b}$ acts on $\mathbb{C}_\lambda = \mathbb{C}$ via $e v = 0$, $h v = \lambda v$ $\forall v \in \mathbb{C}_\lambda$ $\square$

Prop. B: Let $\mathfrak{n}^- = \langle f \rangle \subset \mathfrak{sl}_2$, $\lambda \in \mathbb{C}$ and $v^+ \in \mathbb{C}_\lambda \setminus \{0\}$.

(1) $\mathbb{C}[f] = \mathrm{Sym}(\mathfrak{n}^-) = \mathcal{U}(\mathfrak{n}^-) \longrightarrow M(\lambda)$, $1 \mapsto v^+$

is an isomorphism of $\mathcal{U}(\mathfrak{n}^-)$-modules. In particular $\{ v_j = \frac{1}{j!} f^j v^+ \}_{j \geq 0}$ is a basis of $M(\lambda)$

(2) Let v_0 be a highest weight vector in V of weight λ. Then there is a unique map

$$M(\lambda) \longrightarrow V, \quad v^+ \longmapsto v_0$$

<u>Proof</u> Exercise $\square$

Lecture 5

<u>Thm:</u> Let $\lambda \in \mathbb{C}$. Then

(1) $M(\lambda)$ is irreducible $\iff \lambda \notin \mathbb{Z}_{\geq 0}$

(2) Let $\lambda \in \mathbb{Z}_{\geq 0}$, then there is an injective map

$$M(-\lambda-2) \hookrightarrow M(\lambda) \twoheadrightarrow L(\lambda)$$
$$w^+ \longmapsto \tfrac{1}{(\lambda+1)!} f^{(\lambda+1)} v^+$$

and the quotient is a irr. f.d. representation $L(\lambda)$

of dimension $\lambda+1$. Here w^+, v^+ are the highest weight vectors of $M(-\lambda-2)$ and $M(\lambda)$, respectively.

(3) Let L be a simple, f.d. rep. of $sl_2(\mathbb{C})$. Then there is a (unique up to scalar) highest weight vector $v_0 \in L$. Then v_0 has weight $\lambda \in \mathbb{Z}_{\geq 0}$ and the natural map $M(\lambda) \to L(\lambda)$ yield an iso. $L(\lambda) \xrightarrow{\sim} L$ $\square$

Pf: Denote a basis of $M(\lambda)$ as in Prop. B. Then by Prop A,

(*) $e v_{j+1} = (\lambda - j) v_j = 0 \iff \lambda = j$

(1) "$\Leftarrow$" Assume that $\lambda \notin \mathbb{Z}_{\geq 0}$ and let $x \in M(\lambda) \setminus \{0\}$.

Let $k \in \mathbb{Z}$, s.t. $x = a_k v_k + \sum_{j<k} a_j v_j$ with

$a_k \neq 0$. Then by (*) $e^k x = a v^+$ where $a \neq 0$

Since v^+ generates $M(\lambda)$, so does x. Hence

$M(\lambda)$ is irreducible

(2) + (1) "$\Rightarrow$" If $\lambda \in \mathbb{Z}_{\geq 0}$, by (*) $v_{\lambda+1}$

highest weight vector of (by Prop A) weight

$\lambda - 2(\lambda+1) = -\lambda - 2$. Hence, there is a unique map

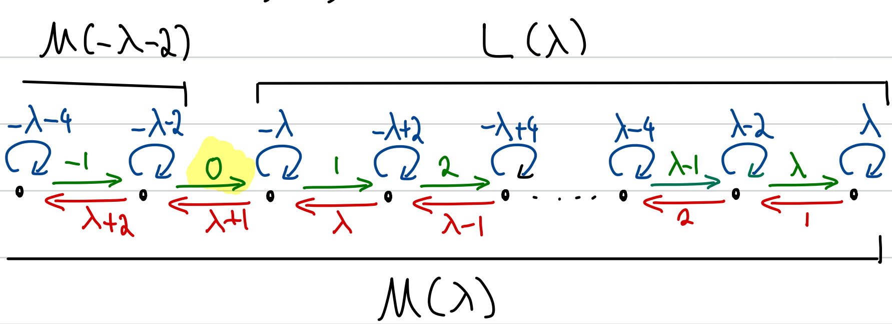
$$M(-\lambda-2) \longrightarrow M(\lambda), \quad w^+ \mapsto v_{\lambda+1}.$$

Since $-\lambda-2 \notin \mathbb{Z}_{\geq 0}$, $M(-\lambda-2)$ is irreducible and the map is hence injective.

By a similar argument as in the proof of (1) "$\Leftarrow$", the quotient $L(\lambda) = M(\lambda)/M(-\lambda-2)$ is irreducible, and $L(\lambda)$ has basis $\bar{v}_0, \ldots, \bar{v}_\lambda$.

(3) Exercise. Hint: decompose L in eigenspaces of h. $\square$

<u>Sketch</u> For $\lambda \in \mathbb{Z}_{\geq 0}$, we obtain:



Example: (1) $L(0) = \mathbb{C}_0$ = trivial

$L(1) = \mathbb{C}^2$ = fundamental

$L(2) = \mathfrak{sl}_2$ = adjoint

In general $L(\lambda) = \mathrm{Sym}^\lambda(\mathbb{C}^2) = k[x,y]_\lambda$

homogeneous polynomials of degree λ $\triangleright$

We will study Lie algebras via copies of $\mathfrak{sl}_2$'s they contain:

Def B: Let $\mathfrak{g}$ be a Lie algebra. A choice of elements

$e, h, f \in \mathfrak{g}$ with $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$

is called an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$

0.5. Nilpotence.

Reminder: let V be vector space with $\dim V = n < \infty$.

Then $x \in \mathrm{End}(V)$ is called nilpotent if (equivalently)

(1) $x^n = 0$ for some $n \geq 0$

(2) $\det(xt - I) = x^{\dim V}$

(3) There is a chain of subspaces

$$\{0\} = V_0 \subset V_1 \subset \ldots \subset V_n$$

with $\dim V_i = i$ and $x V_i \subset V_{i-1}$

(4) V has a basis such that x is a strictly upper-triangular matrix

Moreover x is called triangularizable if (equivalently)

(1) There is a chain

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim V_i = i$ and $x V_i \subset V_i$

(2) There is a basis of V, such that x is upper-triangular $\square$

<u>Def</u> A subset $T \subset \mathrm{End}(V)$ is called triangularizable if there is a chain

$$\{0\} \subset V_0 \subset V_1 \cdots \subset V_n = V \quad \text{with } \dim V_i = 0$$

and $x V_i \subset V_i \quad \forall x \in T$

Thm A (Radjavi) Let $\mathcal{N} \subset \mathrm{End}(V)$ a set of nilpotent operators. Assume that for all $x, y \in \mathcal{N}$ there is a polynomial expression $p(x,y)$ such that

$$xy - p(x,y)x \in \mathcal{N}$$

Then $\mathcal{N}$ is triangularizable.

Proof: Pick a subset $\mathcal{N}_0 \subset \mathcal{N}$ such that the common kernel $\mathcal{K} = \bigcap_{x \in \mathcal{N}_0} \ker x$ has minimal positive dimension. Assume that $\mathcal{N}_0$ is maximal with this property.

Now N_0 acts on V/K which is of smaller dimension. Hence the action of N_0 on V/K is triangularizable by induction. Hence so is N_0.

We show $N = N_0$. Assume the contrary.

Then there is a $y \in N$ with $yK \neq 0$.

Then, K is also not invariant under y, since otherwise y acts nilpotently on K. Then $0 \neq \ker y \cap N_0 \subsetneq K$ which contradicts K minimal.

Hence $yK \neq K$ and hence, we find $x_1 \in \mathcal{N}_0$ with $x_1 y K \neq 0$. Now let $p_1(x_1, y)$ such that

$$y_1 = x_1 y - p_1(x_1, y) \in \mathcal{N}.$$

Then $y_1 K \neq 0$. Hence, again, $y_1 K \neq K$, so we find $x_2 \in \mathcal{N}$ with $x_2 y_1 K \neq 0$ ....

Continuing this way we find $x_1, \dots, x_n \in \mathcal{N}_0$ such that

$$x_n \cdots x_1 y K \neq 0.$$

But, x_i are nilpotent and $\mathcal{N}_0$ is triangularizable.

Hence $x_n \cdots x_1 = 0$. ↯ $\square$

Thm B (Engel) If $\mathfrak{n} \subset \mathrm{End}(V)$ is a Lie algebra of nilpotent operators, then $\mathfrak{n}$ is triangularizable.

Remark: This implies that there is a $0 \neq v \in V$ such that $\mathfrak{n} v = 0$.

Recall that $[U, V] = \langle [u,v] \mid u \in U, v \in V \rangle_{\mathbb{C}}$ for $U, V \subset \mathfrak{g}$.

For example, $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}$ is an ideal, called the derived Lie algebra. Moreover $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is the largest Abelian quotient.

Lecture 6

Def B Let $\mathfrak{g}$ be a Lie algebra. Then we define two sequences of ideals:

(1) The descending central series

$$\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^1 = [\mathfrak{g},\mathfrak{g}],\ \dots,\ \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$$

(2) The derived series

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}],\ \dots,\ \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$$

We call $\mathfrak{g}$

(1) nilpotent if $\mathfrak{g}^i = 0$ for $i \gg 0$,

(2) solvable if $\mathfrak{g}^{(i)} = 0$ for $i \gg 0$.

Example: $\mathfrak{b} = \left\{ \begin{pmatrix} \diagdown & * \\ & \diagdown \end{pmatrix} \right\} \subset \mathfrak{gl}_n(\mathbb{C})$ is solvable

$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & * \\ & \ddots \\ 0 & & 0 \end{pmatrix} \right\} \subset \mathfrak{gl}_n(\mathbb{C})$ is nilpotent.

Moreover $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}$.

Remark (1) Abelian $\Rightarrow$ Nilpotent $\Rightarrow$ solvable

(2) If $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a map of Lie algebras,

then $\varphi(\mathfrak{g}^i) = \varphi(\mathfrak{g})^i$ and $\varphi(\mathfrak{g}^{(i)}) = \varphi(\mathfrak{g})^{(i)}$.

Moreover

$\mathfrak{g}$ is solvable $\Leftarrow$ $\ker\varphi$ and $\operatorname{im}\varphi$ solvable

(3) If $\mathfrak{g} \subset \mathfrak{g}'$ and $\mathfrak{g}'$ is nilpotent, then so is $\mathfrak{g}$

Def C: A Lie algebra $\mathfrak{g}$ is called ad-nilpotent if $\operatorname{ad} x \in \mathfrak{gl}(V)$ is nilpotent for all $x \in \mathfrak{g}$.

Thm C (Engel, v2) Let $\mathfrak{g}$ be finite dimensional. Then

$$\mathfrak{g} \text{ nilpotent} \iff \mathfrak{g} \text{ ad-nilpotent.}$$

Proof: "$\Rightarrow$" Observe that $\operatorname{ad}(x)\, \mathfrak{g}^i \subset \mathfrak{g}^{i+1}$

"$\Leftarrow$" Let $\mathfrak{g}$ be ad-nilpotent. Then $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is triangularizable by Thm B, so there is a basis such that $\operatorname{ad}(\mathfrak{g}) \subset \left(\begin{smallmatrix} & * \\ 0 & \end{smallmatrix}\right)$. Since the elements are nilpotent actually $\operatorname{ad}(\mathfrak{g}) \subset \left(\begin{smallmatrix} 0 & * \\ & \ddots \\ & & 0 \end{smallmatrix}\right)$. So $\operatorname{ad}(\mathfrak{g})$ is nilpotent.

So $(\operatorname{ad} \mathfrak{g})^i = \operatorname{ad} \mathfrak{g}^i = 0$ for $i \gg 0$.

But then $\mathfrak{g}^i \subset \ker(\operatorname{ad}) = \mathfrak{z}(\mathfrak{g}) \Rightarrow \mathfrak{g}^{i+1} = 0$. $\square$

0.6. Solvability - Theorem of Lie

Lemma: Let V be a f.d. rep. of a Lie algebra $\mathfrak{g}$.
Let $I \subset \mathfrak{g}$ be an ideal. Let $\lambda : I \to \mathbb{C}$ be linear form. Then $V_\lambda = \{ w \in V \,/\, xv = \lambda(x) w \;\; \forall x \in I \}$ is a subrepresentation.

Pf: We have to show that $xyw = \lambda(x) yw$ $\forall x \in I, y \in \mathfrak{g}, w \in V_\lambda$. We have always

$$xyw = yxw + [x,y]w = \lambda(x)\, yv + \lambda([x,y])\, w.$$

So, we have to show that $\lambda([x,y]) = 0$ whenever $\exists\, 0 \neq w \in V_\lambda$. Take n maximal such that

$$w, yw, y^2w, \dots, y^n w$$

is linearly independent and let W be the span of these vectors. Now W is stable under y.

Let $W_i = \mathrm{span}(w, yw, \dots, y^i w)$. Hence $yW_{i-1} \subseteq W_i$

We claim that W_i is x-stable.

For $i=0$ $W_0 = \mathrm{span}(w)$ and $xw = \lambda(x) w \in W_0$.

Now $$x y^i w = \underbrace{y \underbrace{(x y^{i-1} w)}_{\in W_{i-1}}}_{\in W_i} + \underbrace{\underbrace{[x,y]}_{\in I} y^{i-1} w}_{\in W_{i-1}} \in W_i$$

by induction. So W_i is x-stable. By a similar induction one can show that

$$x y^i v \in \underbrace{y^i x w}_{= \lambda(x) y^i v} + W_{i-1}.$$

Hence, the matrix of x acting on W has the form

$$\begin{pmatrix} \lambda(x) & & * \\ & \ddots & \\ & & \lambda(x) \end{pmatrix}$$

Hence, $\operatorname{tr}(x|_W) = (\dim W)\,\lambda(x)$.

Now, we apply this to $[x,y] \in I$. Then we get

$$0 = \operatorname{tr}([x,y]|_W) = (\dim W)\,\lambda([x,y])$$

Since $\dim W \neq 0$, we get $\lambda([x,y]) = 0$ $\square$

Remark: This only works in $\operatorname{char} = 0$!

Thm (Lie) Let $0 \neq V$ be f.d. v.s.

Let $g \subset gl(V)$ be a solvable Lie subalgebra.

Then there is a $0 \neq v \in V$ such that $gv \subset \mathbb{C}v$.

Pf: Induction on $\dim g$.

$\dim g = 0$ is clear. Let $\dim g > 0$.

Now, choose any $g \supset I \supset [g,g]$, such that $\dim I = \dim g - 1$. This is possible since $[g,g] \subsetneq g$ since g is solvable.

Since $[g,g]$ is an ideal and $g/[g,g]$ abelian, I is an ideal in g.

By assumption, we find $0 \neq v' \in V$, s.t.

$I v' \subset \mathbb{C} v'$. Hence $x v' = \lambda(x) v'$ $\forall x \in V$.

Now consider V_λ as in the lemma.

Then $v' \in V_\lambda$ so $V_\lambda \neq 0$. Now

$g V_\lambda \subset V_\lambda$. Write $g = I + \mathbb{C} y$.

Choose any eigenvector $0 \neq v \in V_\lambda$ of y.

Then $g v \subset \mathbb{C} v$. $\square$

<u>Cor A</u>: Each simple finite-dimensional rep. of a solvable Lie algebra is 1-dim.

Rem: This is not true for ∞-dim. rep.

See Example 0.1C(B).

Lecture 6

Cor B: Let V be a f.d. rep. of a solvable Lie algebra $\mathfrak{g}$. Then

(1) There is a chain of subrepresentations

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

with $\dim V_i = i$

(2) The image of $\mathfrak{g} \to \mathfrak{gl}(V)$ is triangularizable.

Cor C: If $\mathfrak{g}$ is a f.d. solvable Lie algebra, then $[\mathfrak{g},\mathfrak{g}]$ is nilpotent.

Pf: From Cor. B, we get that $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is triangularizable. Hence $[\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}] = \mathrm{ad}([\mathfrak{g},\mathfrak{g}])$ consists of nilpotent matrices. Since $\ker(\mathrm{ad} : [\mathfrak{g},\mathfrak{g}] \to \mathfrak{gl}(\mathfrak{g})) \subset \mathfrak{z}([\mathfrak{g},\mathfrak{g}])$, $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. $\square$

0.7. Solvability - Cartan's Criterion

<u>Recall</u>: (Jordan - Chevalley)

Let $x \in \mathrm{End}(V)$ for V a f.d. v.s.

Then there are unique $x_s, x_n \in \mathrm{End}(V)$, such that

(1) $x_s + x_n = x$

(2) $x_s x_n = x_n x_s$

(3) x_s is diagonalizable, x_n is nilpotent

Idea: Take Jordan normal form:

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} = \underbrace{\begin{pmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{pmatrix}}_{x_s} + \underbrace{\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}}_{x_n}$$

Moreover, there is a polynomial p, s.t., $x_s = p(x)$

$\square$

Lemma A: Let V be a f.d. v.s.

If $x = x_s + x_n \in End(V)$ is a Jordan-Chevalley decomposition, then so is

$ad\, x = ad(x_s) + ad(x_n) \in End(gl(V))$. So

$$ad(x_s) = (ad(x))_s \quad \text{and} \quad ad(x_n) = (ad(x))_n.$$

Pf: Now $[ad(x_s), ad(x_n)] = ad([x_s, x_n]) = 0$.

The Lie algebra generated by x_n is nilpotent.

(with a correct basis x_n is a strictly upper triangular matrix) so x_n is ad-nilpotent by

0.5. Thm C (Engel $\frac{1}{2}$). Hence $ad(x_n)$ is nilpotent.

To see that ad x_s is nilpotent, let

$\{v_i\}$ be a eigenbasis of x_s acting on V with

eigenvalues λ_i.

Let f_{ij} the map defined by $f_{ij}(v_k) = \delta_{k,i} v_j$.

Then $\{f_{ij}\}$ is a basis of End(V). Now

$$\operatorname{ad} x_i \, f_{ij} = (\lambda_i - \lambda_j) f_{ij}$$

So ad(x_s) is diagonalizable $\square$

Lemma B: Let V be a f.d. v.s., $A \subset B \subset \mathrm{End}(V)$ subspaces and $T = \{x \in \mathrm{End}(V) \mid (\mathrm{ad}(x))(B) \subset A\}$.

Let $x \in T$, such that $\mathrm{tr}(xz) = 0$ $\forall z \in T$, then x is nilpotent.

Pf: Pick such $x \in T$. Write $x = x_s + x_n$ for the Jordan-Chevalley decomposition. We have to show $x_s = 0$. Recall that $x_s = p(x)$. Hence $x_s \in T$.

Since $(\mathrm{ad}(x_s))(B) \subset A$, $\ker(\mathrm{ad}(x_s) - \lambda) \; A \subset \forall \lambda \neq 0$.

Now let $z = \overline{x_s}$. Then, $\ker(\mathrm{ad}(x_s) - \lambda) = \ker(\mathrm{ad}(z) - \lambda) \subset A$ $\forall \lambda \neq 0$. Hence $z \in T$. Now $\mathrm{tr}(x_s z) = \sum a_\lambda |\lambda|^2 = 0$.

Hence $x_s = z = 0$.

<u>Thm</u> (Cartan's solvability criterion v1) Let V be a f.d. v.s. and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie algebra. Then

$$\mathfrak{g} \text{ solvable } \iff tr(xy) = 0 \quad \forall\, x \in [\mathfrak{g},\mathfrak{g}],\ y \in \mathfrak{g}.$$

<u>Proof:</u> "$\Rightarrow$" By 0.6 Cor. B to Lie's theorem, $\mathfrak{g}$ is triangularizable. The rest is clear.

"$\Leftarrow$" It suffices to show that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent.

By 0.5. Thm C, it suffices to show that $\mathfrak{g}$ is ad-nilpotent.

By Lemma A it suffices to show that $x \in [\mathfrak{g},\mathfrak{g}] \subset \mathfrak{gl}(V)$ is nilpotent as an endomorphism of V.

Using Lemma B, it suffices to show that

$$\operatorname{tr}(xz) = 0 \quad \text{for all } z \text{ with } [z, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]$$

Write $x = \sum [c_i, d_i]$, then

$$\operatorname{tr}(xz) = \sum \operatorname{tr}([c_i, d_i] z) = \sum \operatorname{tr}(\underbrace{c_i}_{\in \mathfrak{g}} \underbrace{[d_i, z]}_{\in [\mathfrak{g}, \mathfrak{g}]}) = 0$$

by assumption. $\square$

0.8 Killing form

Let $\mathfrak{g}$ be a Lie algebra

<u>Def</u>: (1) A symmetric form $B: \mathfrak{g} \otimes \mathfrak{g} \to k$ is called <u>$\mathfrak{g}$-invariant</u> if

$$B([x,y],z) = B(x,[y,z]) \quad \forall x,y,z \in \mathfrak{g}$$

(2) If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is a f.d. rep. of $\mathfrak{g}$, we denote

$$(x,y)_V = \mathrm{tr}(\rho(x)\rho(y))$$

If $\rho = ad$, $(\ ,\ ) = (\ ,\ )_V$ is the <u>Killing form</u>

(3) The radical of a $\mathfrak{g}$-invariant form B is

$$\mathrm{rad}\, B = \{x \in \mathfrak{g} \mid B(x,y) = 0 \ \forall y \in \mathfrak{g}\}.$$

Lemma: (1) The form $(\, , \,)_V$ defined above is $\mathfrak{g}$-invariant.

(2) rad B is an ideal

(3) Let $I \subset \mathfrak{g}$ be an ideal. Then the Killing form of I is the Killing form of $\mathfrak{g}$ restricted to I

Pf: (1) + (2) Exercise

(3) In general, let $a \in \mathrm{End}(V)$, $W \subset V$ s.t. $a(W) \subset W$. Then $\mathrm{tr}\, a = \mathrm{tr}\, a|_W$.

Now apply this to $V = \mathfrak{g}$, $W = I$, $a = \mathrm{ad}\,x\, \mathrm{ad}\,y$ for $x, y \in I$. $\square$

Thm (Cartan's solvability criterion v2) Let $\mathfrak{g}$ be a f.d. Lie algebra. Then

$$\mathfrak{g} \text{ solvable} \iff \mathfrak{g} \perp [\mathfrak{g},\mathfrak{g}]$$

where $\perp$ is meant with respect to the Killing form.

Proof: The Cartan criterion v1 shows that

$$\mathfrak{g} \perp [\mathfrak{g},\mathfrak{g}] \iff (x,[y,z]) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}([x,y])) = 0 \quad \forall x,y,z \in \mathfrak{g}$$

$$\iff \operatorname{ad}(\mathfrak{g}) \text{ is solvable}.$$

Now use the s.e.s. $\mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \operatorname{ad}(\mathfrak{g})$ $\square$

Example: (1) Let $\mathfrak{g} = \mathfrak{sl}_2$ (with the basis e, h, f)

Recall that

$$ad(e) = \begin{pmatrix} 0 & -2 & 0 \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad ad(h) = \begin{pmatrix} 2 & & \\ & 0 & \\ & & -2 \end{pmatrix} \quad ad(f) = \begin{pmatrix} 0 & & \\ -1 & 0 & \\ 0 & 2 & 0 \end{pmatrix}$$

Hence, the Killing form has matrix

$$\begin{array}{c|ccc} & e & h & f \\ \hline e & 0 & 0 & 4 \\ h & 0 & 8 & 0 \\ f & 4 & 0 & 0 \end{array}$$

Let $V = \mathbb{C}^2$ be the fundamental representation, so that $(x, y)_V = \operatorname{tr}(xy)$. Then the corr. form has matrix

$$\begin{pmatrix} & & 1 \\ & 2 & \\ 1 & & \end{pmatrix}$$

(3) Let $\mathfrak{g} = \langle u, v, z \rangle$ be the Heisenberg algebra. Then $(\, , \,) = 0$.

Lecture 7

0.9. Semisimple and reductive Lie algebras

Def A: Let $\mathfrak{g}$ be a Lie algebra, then

(1) $\mathfrak{g}$ is called simple if $\{0\}, \mathfrak{g}$ are the only subalgebras, and $\mathfrak{g}$ is not abelian (So $\{0\}$ is not simple)

(2) $\mathfrak{g}$ is called semisimple if it is isomorphic to a finite direct sum of f.d. simple Lie algebras.

(So $\{0\} = \bigoplus_{\emptyset}$ is semisimple)

(3) $\mathfrak{g}$ is called reductive if it is a direct sum of a f.d. Abelian Lie algebra with a a semisimple Lie algebra

Thm: Let $\mathfrak{g}$ be a f.d. Lie algebra. Then, the following are equivalent:

(1) $\mathfrak{g}$ has no non-zero Abelian ideal

(2) ——— " ——— solvable ideal

(3) $\mathfrak{g}$ is semisimple

(4) $\mathfrak{g}$ is a direct sum of its simple ideals

(5) the Killing form of $\mathfrak{g}$ is non-degenerate

Pf: $4 \Rightarrow 3 \Rightarrow 2 \Leftrightarrow 1$ is clear.

"$2 \Rightarrow 5$" The Killing form is non-deg. $\Leftrightarrow$ $R = \mathrm{rad}(\,,\,) = 0$.

By 0.8 lemma, $R \subset \mathfrak{g}$ is an ideal and the

Killing form of R is trivial. So by Cartan's solvability criterion v2, $R=0$

"$5 \Rightarrow 1$" Let $I < \mathfrak{g}$ be an Abelian ideal. Then $((\mathrm{ad}\,x)(\mathrm{ad}\,y))^2 = 0$ $\forall x \in \mathfrak{g}, y \in I$. Hence, $(\mathrm{ad}\,x)(\mathrm{ad}\,y)$ is nilpotent and $(x,y)=0$ for all $x \in \mathfrak{g}, y \in I$. Hence $I \subset \mathrm{rad}(,)$.

"$2 \Rightarrow 4$" Assume that $\mathfrak{g}$ has no non-zero solvable ideals. Let $I \subset \mathfrak{g}$ be an ideal. Then $I^\perp \subset \mathfrak{g}$ is also an ideal. Then also $I \cap I^\perp$ is an ideal, on which the Killing form vanishes. Hence, by Cartan's solvability criterion, $I \cap I^\perp$ is solvable. Hence $0 = I \cap I^\perp$. Now

$[I, I^\perp] \subset I \cap I^\perp = 0$, so I and $I^\perp$ commute. Hence

$\mathfrak{g} = I \oplus I^\perp$ as Lie algebra. Ideals in I (and $I^\perp$)

are also ideals in $\mathfrak{g}$, so that $I, I^\perp$ have no non-zero

solvable ideals, too. By induction, we may hence write

$$\mathfrak{g} = I_1 \oplus \cdots \oplus I_r \quad \text{as a sum of simple ideals.}$$

If $I \subset \mathfrak{g}$ is any simple ideal, then

$$I = [I, \mathfrak{g}] = [I, I_1] \oplus \cdots \oplus [I, I_m],$$

so $I = I_k$ for some k $\square$

Rem: (1) If $\mathfrak{g}$ is reductive, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{z}(\mathfrak{g})$,

where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple

(2) Ideals and quotients of semisimple (reductive) Lie algebras

are also semisimple (reductive)

3.10 The Casimir operator

Let $\mathfrak{g}$ be a f.d. Lie algebra.

<u>Def</u> Let B be a non.deg. $\mathfrak{g}$-invariant bilinear form.

Then we define

$$C_B = \sum x_i x^i \in \mathcal{U}(\mathfrak{g})$$

where $\{x_i\}$, $\{x^i\}$ are dual bases of $\mathfrak{g}$ (so $B(x_i, x^j) = \delta_{ij}$)

<u>Lemma</u> (1) The element C_B does not depend on a choice of basis. Moreover, $C_B \in Z(\mathcal{U}(\mathfrak{g}))$.

(2) Let $0 \neq V$ be a f.d. rep. of a semisimple Lie algebra. Then $(\, , \,)_V$ is non-deg.

Proof: (1) B induces an iso. $\mathfrak{g} \to \mathfrak{g}^*$ and a map of $\mathfrak{g}$-rep's:

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^* = \mathfrak{gl}(\mathfrak{g})$$
$$C_B \longmapsto \mathrm{Id}_{\mathfrak{g}}$$

So C_B does not depend on a choice and commutes with $\mathfrak{g}$-action (since $\mathrm{Id}_{\mathfrak{g}}$ does)

(2) Exercise: Use Cartan criterion on rad $(\,,\,)$ $\square$

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$, $B(X,Y) = \mathrm{tr}(XY)$, then we get dual bases $\{e, h, f\}$, $\{f, \frac{1}{2}h, e\}$ and

$$C = C_B = ef + \tfrac{1}{2}h^2 + fe = h(h+2)/2 + 2fe$$

0.11 Homomorphisms and invariants

<u>Def</u>: (1) For $\mathfrak{g}$-representations V, W, we denote

$$\mathrm{Hom}_{\mathfrak{g}}(V,W) = \{ \varphi \in \mathrm{Hom}_{\mathbb{C}}(V,W) \mid \varphi(xv) = x\varphi(v) \ \forall x \in \mathfrak{g}, v \in V \}$$

and similarly $\mathrm{End}_{\mathfrak{g}}(V) = \mathrm{Hom}_{\mathfrak{g}}(V,V)$

(2) For a $\mathfrak{g}$-rep V, we denote

$$V^{\mathfrak{g}} = \{ v \in V \mid xv = 0 \ \forall x \in \mathfrak{g} \}$$

the $\mathfrak{g}$-invariants of V.

<u>Remark</u>: (1) For $\mathfrak{g}$-rep's V, W there is a natural $\mathfrak{g}$-action on $\mathrm{Hom}_{\mathbb{C}}(V,W)$ via

$$(x\varphi)(v) = x\varphi(v) - \varphi(xv)$$

This way, we obtain

$$\mathrm{Hom}_{\mathfrak{g}}(V,W) = \mathrm{Hom}_{\mathbb{C}}(V,W)^{\mathfrak{g}}$$

(2) For all representations V, there is a natural map

$$Z(U(\mathfrak{g})) \to \mathrm{End}_{\mathfrak{g}}(V), \quad z \mapsto (v \mapsto zv)$$

In particular, this applies to Casimir operators! $\square$

Lecture 8

<u>Lemma</u> (Schur) Let L be a simple f.d. rep. of $\mathfrak{g}$.

Then $\mathrm{End}_{\mathfrak{g}}(L) = \mathbb{C}\, \mathrm{id}_L$

Pf: Let $\varphi \in \mathrm{End}_{\mathfrak{g}}(L)$. Since $L \neq 0$, φ has at least one eigenvalue, say λ. Now the eigenspace $0 \neq L_\lambda \subset L$ is a subrepresentation. Since L is simple, $L = L_\lambda$ and $\varphi = \lambda$ $\square$

Example: (i) Let $g = \mathfrak{sl}_2$ and $L(n)$ the $n+1$-dim simple rep. with highest weight vector v^+.

Consider the Casimir

$$C = \tfrac{1}{2}h(h+2) + fe \quad \in Z(U(\mathfrak{sl}_2))$$

$$\begin{aligned} Cv^+ &= \left(\tfrac{1}{2} h(h+2) + fe\right) v^+ \\ &= \tfrac{1}{2} h(h+2)v^+ + fe\, v^+ \\ &= \tfrac{1}{2} n(n+2)\, v^+ \end{aligned}$$

Hence C acts on $L(n)$ via multiplication with $\frac{1}{2}n(n+2)$. We see that we can decompose any f.d. $\mathfrak{sl}_2$-rep. as $V = \bigoplus \underbrace{L(n)}_{\frac{1}{2}n(n+2)\text{ - Eigenspace of } C}{}^{m_n}$

(2) Let $\mathfrak{g} \subset \mathfrak{gl}(V)$. Then $C = C_V = C_{(\,,\,)_V}$ acts as the identity on V, so $C \mapsto 1 \in \mathrm{End}(V)$.

In particular, $\mathrm{tr}(C) = \dim V$ $\square$

<u>Thm:</u> Let V be a f.-d. rep. of a semisimple Lie algebra $\mathfrak{g}$. Then $V = V^{\mathfrak{g}} \oplus \mathfrak{g}V$

Pf: Wlog. $\mathfrak{g} < \mathfrak{gl}(V)$ (else proceed with $\mathrm{im}(\mathfrak{g} \to \mathfrak{gl}(V))$).

We prove this by induction on $\dim V$.

If $V = V^{\mathfrak{g}}$, we are done. Assume $V \neq V^{\mathfrak{g}}$.

As in Example (2) denote by C the Casimir.

Then we can decompose V into the generalized eigenspaces of C. These are subrepresentations.

If C has more than one eigenvalue, the different eigenspaces have a smaller dimension than V and we can proceed by induction.

If C has just one eigenspace for $\lambda \neq 0$, then since $\operatorname{tr}(C) = \lambda \dim V$ the eigenvalue is not 0.

Hence $V = CV$ and hence $V = \mathfrak{g}V = \mathfrak{g}V \oplus \underbrace{V^{\mathfrak{g}}}_{=0}$ $\square$

0.12 Weyl's theorem

Thm (Weyl): Each f.d. rep. of a semisimple Lie algebra is semisimple

Pf: Let $\mathfrak{g}$ be s.s. and V a f.d. rep. Let $U \subset V$ be a subrep. Then we obtain a surjection

$$\mathrm{Hom}(V,U) \twoheadrightarrow \mathrm{Hom}(U,U) = \mathrm{End}(U)$$

using 0.11 Thm, this yields a surjection

$$\begin{array}{ccc} \mathrm{Hom}(V,U)^{\mathfrak{g}} & \twoheadrightarrow & \mathrm{End}(U)^{\mathfrak{g}} \\ \| & & \| \\ \mathrm{Hom}_{\mathfrak{g}}(V,U) & \twoheadrightarrow & \mathrm{End}_{\mathfrak{g}}(U) \end{array}$$

Denote by p a preimage of $\mathrm{id}_U \in \mathrm{End}_{\mathfrak{g}}(U)$. Then $V = U \oplus \ker p$

By Schur's Lemma C act by a scalar on $L(\lambda)$. Let $v^+ \in L(\lambda)$ the highest weight vector, then $Cv^+ = (\frac{1}{2}h^2 + h + fe)\, v^+ = (\frac{1}{2}\lambda^2 + \lambda) \cdot v^+$

Since $\lambda \mapsto \frac{1}{2}\lambda^2 + \lambda$ is injective for $\lambda \in \mathbb{Z}_{\geq 0}$, we can decompose f.d. sl_2-rep's in isotypic components using eigenspaces of C $\square$

Cor A Let $\mathfrak{g}$ be finite dimensional. Then $\mathfrak{g}$ is reductive if and only if ad is a semisimple representation.

0.13 Jordan decomposition in semisimple Lie algebras

<u>Lemma A</u>: Let $I \subset \mathfrak{g}$ be a semisimple ideal in a f.d. Lie algebra. Then $\mathfrak{g} = I \oplus I^{\perp}$.

<u>Proof</u>: $I \cap I^{\perp} \subset I$ is an ideal so also semisimple. But $I \cap I^{\perp} \subset \mathrm{rad}(\ )_I = 0$.

<u>Lemma B</u>: Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ a semisimple subalgebra where V is f.d. Let $x = x_s + x_n$ a Jordan-Chevalley decomposition of $x \in \mathfrak{g}$ with $x_s, x_n \in \mathfrak{gl}(V)$. Then $x_s, x_n \in \mathfrak{g}$

Proof: Denote

$$I \subset \{ y \in gl(V) \mid [y, g] \subset g, \text{ for each } W \subset V \text{ } g\text{-stable } yW \subset W \text{ and } tr(y|_W) = 0 \}$$

Since y_s, y_n are polynomials in y, we have

$$y \in I \Leftrightarrow y_s \in I, y_n \in I$$

So we need to show $g = I$. Since $g = [g, g]$, we have $tr(y|_W) = 0$ $\forall y \in g$. Hence $g \subset I$.

Write $I = g \oplus g^{\perp}$ as in lemma A, so that $[g, g^{\perp}] = 0$. Write $V = \bigoplus W_i$ where W_i are simple. Then $g^{\perp}$ acts on W_i via elements in $\mathrm{End}_{\mathfrak{g}}(V) = \mathbb{C}$, so by scalar.

But since $\mathrm{tr}(y|_{W_i}) = 0$, $y|_{W_i} = 0$ $\forall y \in \mathfrak{g}^{\perp}$.

Hence $y = 0$. Hence $\mathfrak{g}^{\perp} = 0$ and $I = \mathfrak{g}$ ∎

<u>Thm</u> (Absolute Jordan–Chevalley decomposition)

Let $\mathfrak{g}$ be a semisimple Lie algebra.

(1) Each $x \in \mathfrak{g}$ has a unique decomposition $x = s + n$ with

$s, n \in \mathfrak{g}$, $[s, n] = 0$, $\mathrm{ad}\, s$ diagonalizable, $\mathrm{ad}\, n$ nilpotent.

We call this the absolute Jordan-Chevalley decomposition.

(2) Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ any f.d. representation.

Then $\rho(s) = \rho(x)_s$, $\rho(n) = \rho(x)_n$.

(3) Let $\phi : g \to g'$ any homomorphism of semisimple Lie algebras. Then $\phi(x) = \phi(s) + \phi(x)$ is the absolute Jordan-Chevalley decomposition of $\phi(x)$.

Pf: (1) Write $ad\,x = (ad\,x)_s + (ad\,x)_n$ in $gl(g)$. Using Lemma B for $ad(g) \subset gl(g)$ shows that $(ad\,x)_s, (ad\,x)_n \in ad(g)$. Take $s, n \in g$ such that $ad\,s = (ad\,x)_s$ and $ad\,n = (ad\,x)_n$. This shows the existence. Uniqueness follows from uniqueness of Jordan-Chevalley decomposition of $ad\,x$ in $gl(g)$.

Lecture 9

(2) Consider the commutative diagram.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, two heads] \arrow[d, "\mathrm{ad}_{\mathfrak{g}} x"'] & \rho(\mathfrak{g}) \arrow[r, hook] \arrow[d, "\mathrm{ad}_{\rho(\mathfrak{g})} \rho(x)"] & \mathfrak{gl}(V) \arrow[d, "\mathrm{ad}_{\mathfrak{gl}} \rho(x)"] \\ \mathfrak{g} \arrow[r, two heads] & \rho(\mathfrak{g}) \arrow[r, hook] & \mathfrak{gl}(V) \end{tikzcd}$$

The diagram also commutes when replacing the vertical arrows by their semisimple part (this is a general fact on the functoriality of the Jordan-Chevalley decomposition)

Now by (1), $(\mathrm{ad}_{\mathfrak{g}}(x))_s = \mathrm{ad}_{\mathfrak{g}}(s)$ so that also

$$(\mathrm{ad}_{\rho(\mathfrak{g})}(\rho(s)))_s = (\mathrm{ad}_{\rho(\mathfrak{g})}\rho(x))_s = \mathrm{ad}_{\rho(\mathfrak{g})}(\rho(x)_s)$$

Since $\mathrm{ad}_{\rho(\mathfrak{g})} : \rho(\mathfrak{g}) \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ is an injection

$\rho(s) = \rho(x)_s$

(3) Exercise $\square$

<u>Remark</u>: We call $x \in \mathfrak{g}$ ad-nilpotent or ad-semisimple if $\operatorname{ad} x$ is nilpotent or diagonalizable, resp.

For semisimple Lie algebras, we simply say nilpotent or diagonalizable. Also we don't distinguish the different versions of Jordan Chevalley.

I. Semisimple Lie algebras

I.1 Cartan subalgebras and roots

Let $\mathfrak{g}$ be a semisimple Lie algebra.

<u>Def</u> A: We call $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra,

if (1) $\mathfrak{h}$ is a maximal Abelian subalgebra

(2) h is semisimple $\quad \forall h \in \mathfrak{h}$ $\square$

<u>Rem</u>: All Cartan subalgebras are conjugate wrt.

$\mathrm{Int}(\mathfrak{g})$. We will (maybe) show this later $\square$

From now, choose a Cartan $\mathfrak{h} \subset \mathfrak{g}$

<u>Def B</u>: (1) For a rep. V of of $\mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$,

we call

$$V_\lambda = \{ v \in V \mid hv = \lambda(h) v \ \forall h \in \mathfrak{h} \}$$

the <u>weight space</u> of V (wrt. λ). If $V_\lambda \neq 0$,

we call λ a <u>weight</u> of V

(2) We call $0 \neq \alpha \in \mathfrak{h}^*$ a <u>root</u> of $\mathfrak{g}$ if it

is a weight of the adjoint rep. $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$

and denote by $\Phi \subset \mathfrak{h}^*$ the set of roots

(3) We denote by

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

the weight space decomposition of $\mathfrak{g}$ $\square$

<u>Example</u>: Let $\mathfrak{g} = \mathfrak{sl}_n$. Then

$$\mathfrak{h} = \{ \mathrm{diag}(a_1, \dots, a_n) \mid \sum a_i = 0 \} \subset \mathfrak{sl}_n$$

is a Cartan subalgebra. Write

$$\varepsilon_i : \mathfrak{h} \to \mathbb{C}, \quad \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i .$$

Then $\mathfrak{h}^* = \mathrm{span}(\varepsilon_i) \cong \mathbb{C}^n / (1, \dots, 1) \cong \mathbb{C}^{n-1}$.

Denote by E_{ij} the matrix with a 1 in position (i,j).

Then $\quad [h, E_{i,j}] = (\varepsilon_i - \varepsilon_j)(h) E_{ij}$.

Write $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$. Then $\Phi = \{ \alpha_{i,j} \mid i \neq j, 1 \leq i,j \leq n \}$

and $\quad \mathfrak{g} = \mathfrak{sl}_n = \underset{\substack{\| \\ \mathfrak{h}}}{\mathfrak{g}_0} \oplus \bigoplus_{i \neq j} \underset{\substack{\| \\ \mathbb{C} E_{ij}}}{\mathfrak{g}_{\alpha_{ij}}}$

Remark: For $\alpha, \beta \in \Phi$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $h \in \mathfrak{h}$, we get

$$[h,[x,y]] = [[h,x],y] + [x,[h,y]] = (\alpha+\beta)(h)\,[x,y]$$

$$([x,y],h) = ([x,h],y) = (x,[h,y])$$

$$= -\alpha(h)\,(x,y) = \beta(h)\,(x,y)$$

Some immediate consequences:

Lemma: Let $\alpha, \beta \in \Delta$.

(1) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$

(3) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\beta \neq -\alpha$

(4) $(\mathfrak{g}_\alpha, \mathfrak{g}_0) = 0$

(5) $(\, , \,)_{\mathfrak{g}_0 \times \mathfrak{g}_0}$ is non-deg.

(6) $(\, , \,)_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-deg pairing

(6) $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$

(7) $\bigcap_{\alpha \in \Delta} \ker \alpha \subset \mathfrak{z}(\mathfrak{g}) = 0$

(8) $\mathfrak{h}^* = \langle \alpha \in \Phi \rangle_{\mathbb{C}}$ $\square$

Thm A: The Cartan subalgebra is self-centralizing, so

$$\mathfrak{h} = \mathfrak{g}_0 = \{x \in \mathfrak{g} \mid [h, x] = 0 \ \forall h \in \mathfrak{h}\}.$$

Pf: let $x \in \mathfrak{g}_0$. Write $x = s + n$ for the Jordan-Chevalley decomposition. Since $\operatorname{ad} x|_{\mathfrak{h}} = 0$ it follows that $(\operatorname{ad} x)_s|_{\mathfrak{h}} = 0 = (\operatorname{ad} x)_n|_{\mathfrak{h}} = 0$. Since $(\operatorname{ad} x)_s = \operatorname{ad} s$ and $(\operatorname{ad} x)_n = \operatorname{ad} n$, we get $s, n \in \mathfrak{g}_0$.

The sum of commuting semisimple elements is also semisimple, hence $s \in \mathfrak{h}$. Hence, $\operatorname{ad} x|_{\mathfrak{g}_0} = \operatorname{ad} n|_{\mathfrak{g}_0}$ is nilpotent.

Hence x is ad-nilpotent in $\mathfrak{g}_0$ for all $x \in \mathfrak{g}_0$.

Hence by Theorem of Engel v2, (O.S.ThmC) $\mathfrak{g}_0$ is nilpotent.

Using the Lie's Theorem (0.6 Thm), hence ad $\mathfrak{g}_0$ is triangularizable. Hence, if ad z is nilpotent for $z \in \mathfrak{g}_0$, then ad z is a strictly upper-triangular matrix. Hence $(z, \mathfrak{g}_0) = \mathrm{tr}(\mathrm{ad}\, z \ \mathrm{ad}\, \mathfrak{g}_0) = 0$.

Hence $z = 0$ by Lemma (5). Hence $\mathfrak{g}_0 = \mathfrak{h}$ $\square$

Rem: We hence get $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$,

$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$ and $(,)|_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate $\square$

Lecture 10

Thm B: For $\alpha \in \Phi$, $\dim [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$ and

$\alpha([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = \mathbb{C}$

Proof: let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, $h \in \mathfrak{h}$. Then

$(h,[x,y]) = ([h,x],y) = \alpha(h)(x,y)$. Since $(\ )|_{\mathfrak{h}\times\mathfrak{h}}$ is non-deg. we get

$$\mathfrak{h} = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \underbrace{[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]^\perp \cap \mathfrak{h}}_{\supset \ker(\alpha)}.$$

Hence $\dim [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \dim \mathfrak{h} - \dim [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]^\perp \cap \mathfrak{h}$

$$\leq \dim \mathfrak{h} - \dim \ker(\alpha) = 1.$$

Now, choose $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, $h \in \mathfrak{h}$ such that $(x,y) \neq 0$ and $\alpha(h) \neq 0$. Then $(h,[x,y]) \neq 0$ so $[x,y] \neq 0$. Hence $\dim [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = 1$.

Now let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ with $h = [x,y] \neq 0$.

Assume that $\alpha(h) = 0$. But then x, y, h span a nilpotent Lie algebra (the Heisenberg algebra).

Then $\operatorname{ad} x$, $\operatorname{ad} y$ and $\operatorname{ad} h$ are triangularizable.

But then $\operatorname{ad} h = \operatorname{ad}[x,y]$ is nilpotent. Since $h \in \mathfrak{h}$ is semisimple by assumption $h = 0$ $\lightning$.

Hence $\alpha(h) = \alpha([x,y]) \neq 0$ $\square$

I.2 Summoning sl_2

<u>Thm</u>: For each $\alpha \in \Phi$, let

$$s_\alpha = \mathfrak{g}_\alpha \oplus [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_{-\alpha}.$$

Then there is an isomorphism $sl_2 \cong s_\alpha$ and an sl_2-triple

$e_\alpha, h_\alpha = \alpha^\vee, f_\alpha \in s_\alpha$, s.t., $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$, $\mathfrak{g}_{-\alpha} = \mathbb{C} e_{-\alpha}$.

Moreover, h_α is uniquely determined by $\alpha(h_\alpha) = 2$ and $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ $\square$

<u>Proof</u>: Sketch: Pick $\tilde{f}_\alpha \in \mathfrak{g}_\alpha$, $\tilde{x}_\alpha \in \mathfrak{g}_{-\alpha}$, s.t.

$(\tilde{e}_\alpha, \tilde{f}_\alpha) = 1$ and $\tilde{h}_\alpha = [e_\alpha, f_\alpha] \neq 0$ which is

possible by by 1.1 Thm B. Now we set

$$h_\alpha = \frac{2}{\alpha(\tilde{h}_\alpha)} \tilde{h}_\alpha, \quad e_\alpha = \tilde{e}_\alpha \quad \text{and} \quad f_\alpha = \frac{2}{\alpha(\tilde{h}_\alpha)} \tilde{f}_\alpha$$

Now $[e_\alpha, f_\alpha] = h_\alpha$, and since $\alpha(h_\alpha) = 2$,

$[h_\alpha, e_\alpha] = \alpha(h_\alpha) e_\alpha = 2 e_\alpha$ and $[h_\alpha, f_\alpha] = -\alpha(h_\alpha) f_\alpha = -2 f_\alpha$.

Hence $e_\alpha, h_\alpha, f_\alpha$ form an $\mathfrak{sl}_2$-triple.

Let $\mathfrak{s}_\alpha' = \langle e_\alpha, h_\alpha, f_\alpha \rangle_{\mathbb{C}} \cong \mathfrak{sl}_2$. Now $\mathfrak{s}_\alpha'$ acts via

the adjoint rep on

$$V = \mathbb{C} h_\alpha \oplus \bigoplus_{t \neq 0} \mathfrak{g}_{t\alpha}$$

where we use that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} h_\alpha$. (I.1 Thm B).

Denote by V_a the eigenspace of h_α on V with eigenvalue a.

Since $\alpha(h_\alpha) = 2$, $V_a = \mathfrak{g}_{\frac{1}{2}\alpha}$ and $V_0 = \mathbb{C} h_\alpha$.

By our description of f.d. representations of $\mathfrak{sl}_2$, we know that $V = V_{even} \oplus V_{odd}$ where

$$V_{even} = \bigoplus_{a \in 2\mathbb{Z}} V_a = \bigoplus_{n \in 2\mathbb{Z}} L(n)^{m_n} \quad V_{odd} = \bigoplus_{a \in 2\mathbb{Z}+1} V_a .$$

Since $L(n)_0 = \mathbb{C}$ for all $n \in \mathbb{Z}$, and $V_0 = \mathbb{C}$, it follows that $V_{even} \cong L(2)$. But $L(2) = S_\alpha' \subset V_{even}$.

Hence $V_{even} = S_\alpha' = S_\alpha$. $\square$

Def: The elements $\alpha^\vee = h_\alpha \in \mathfrak{h}$ are called <u>coroots</u>. The set of coroots is denoted by

$$\Phi^\vee = \{\alpha^\vee \in \mathfrak{h}^* \mid \alpha \in \Phi\} \qquad \square$$

<u>Rem</u>: By the theorem, there is a bijection $\Phi \Leftrightarrow \Phi^\vee$, $\alpha \mapsto \alpha^\vee$ $\square$

In the following, we will write

$$\langle\, , \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C} \qquad \langle \lambda, h \rangle = \lambda(h)$$

for the natural pairing.

I.3. Adjoint action of $\mathfrak{s}_\alpha$ on $\mathfrak{g}$.

For roots $\alpha, \beta \in \Delta$, we consider

$$V_{\alpha,\beta} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \subset \mathfrak{g}.$$

Now $V_{\alpha,\beta}$ is a f.d. representation of $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$

Let p, q be minimal, resp. maximal, s.t.

$\mathfrak{g}_{\beta + p\alpha} \neq 0$ and $\mathfrak{g}_{\beta + q\alpha} \neq 0$.

In particular $p \leq 0 \leq q$

Thm: (1) $p+q = -\beta(h_\alpha) = -\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$

(2) $\{\beta + j\alpha \mid j \in \mathbb{Z}\} \cap (\Phi \cup \{0\}) = \{\beta + j\alpha \mid p \leq j \leq q\}$

(3) $\beta + (p+q)\alpha = \beta - \langle \beta, \alpha \rangle \alpha^\vee \in \Phi$

(4) If $\alpha \pm \beta \neq 0$, then

(a) if $\alpha + \beta \notin \Phi$, $\langle \beta, \alpha^\vee \rangle \geq 0$

(b) if $\alpha - \beta \notin \Phi$, $\langle \beta, \alpha^\vee \rangle \leq 0$

(5) $[s_\alpha, s_\beta] = 0 \iff \langle \beta, \alpha^\vee \rangle = 0 \iff \langle \alpha, \beta^\vee \rangle = 0$

(6) If $\alpha + \beta \neq 0$, $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$

(7) $\mathbb{C}\alpha \cap \Phi = \{\pm \alpha\}$

Proof: (1), (2), (3) $h_\alpha \in \mathfrak{s}_\alpha$ acts on $\mathfrak{g}_{\beta+i\alpha}$ via

$$(\beta+i\alpha)(h_\alpha) = \beta(h_\alpha) + 2i.$$

So, we obtain a $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2$ rep. $V_{\alpha,\beta}$ with h_α action

$$\mathfrak{g}_{\beta+p\alpha} \quad \cdots \quad \mathfrak{g}_{\beta+i\alpha} \quad \cdots \quad \mathfrak{g}_{\beta+q\alpha}$$

$$\circlearrowright \qquad\qquad\qquad\qquad\qquad\qquad \circlearrowright$$

$$\beta(h_\alpha)+2p \qquad\qquad\qquad\qquad \beta(h_\alpha)+2q$$

From f.d. rep. th. of $\mathfrak{sl}_2$, we know that the extremal weights differ by a sign, so that

$$\beta(h_\alpha) + 2p = -(\beta(h_\alpha) + 2q)$$

and hence

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) = -(p+q) \in \mathbb{Z}.$$

Moreover, $g_{\beta+i\alpha} \neq 0$ for all $p \leq i \leq q$.

In particular for $i = p+q$.

(4)-(7) : Exercise $\square$

<u>Def</u>: We call the sequence

$$\beta + p\alpha, \ldots, \beta + q\alpha$$

the α-string through β $\square$

I.4 Root system and Cartan matrix

Let $\mathfrak{h} \subset \mathfrak{g}$ be a semisimple Lie algebra with Cartan.

So far, we defined

(1) Roots $\quad \alpha \in \Phi \subset \mathfrak{h}^*$

(2) Coroots $\quad h_\alpha = \alpha^\vee \in \Phi^\vee \subset \mathfrak{h}$

(3) A bijection $\quad \Phi \to \Phi^\vee, \ \alpha \mapsto \alpha^\vee$

We denote the natural pairing

$$\langle , \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}.$$

and for $A \subset \mathbb{C}$ a subring

$$\mathfrak{h}_R = \sum_{\alpha^\vee \in \Delta^\vee} A\alpha^\vee, \quad \mathfrak{h}^*_R = \sum_{\alpha \in \Delta} A\alpha$$

By 0.9. Prop (1), we have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) \in \mathbb{Z} \qquad \forall \alpha, \beta \in \Delta$$

So the pairing descends, for $R \subset \mathbb{C}$, to

$$\langle \, , \, \rangle : \mathfrak{h}_R \times \mathfrak{h}^*_R \to R.$$

We now collect all the properties of roots and coroots in a convenient framework.

Def: An abstract root system is a tuple

$$(V, \Phi, \Phi^\vee, (\ )^\vee), \quad \text{such that}$$

(1) V is a real vector space, generated by Φ.

(2) $\Phi \subset V$, $\Phi^\vee \subset V^\vee$ are finite subsets and

$(\ )^\vee : \quad - \quad {}^\vee$ a bijection

(3) For $\alpha, \beta \in \Delta$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$ and $\langle \alpha, \alpha^\vee \rangle = 2$

(4) $s_\alpha : V \to V$, $v \mapsto v - \langle v, \alpha^\vee \rangle \alpha$

permutes Φ.

(5) If α and $c\alpha \in \Phi$, then $c = \pm 1$

Thm $(\mathfrak{h}^*_{\mathbb{R}}, \Phi, \Phi^\vee, (\ )^\vee)$ is an abstract root system

Proof: See 0.3 Thm $\square$

I.5 The Killing form on the Cartan.

Recall the proof of I.2 thm. For $\alpha \in \Phi$, we picked $\tilde{e}_\alpha \in \mathfrak{g}_\alpha$, $\tilde{f}_\alpha \in \mathfrak{g}_{-\alpha}$, s.t. $(\tilde{e}_\alpha, \tilde{f}_\alpha) = 1$ and set $\tilde{h}_\alpha = [\tilde{e}_\alpha, \tilde{f}_\alpha]$. Then

$$(\tilde{h}_\alpha, h) = ([\tilde{e}_\alpha, \tilde{f}_\alpha], h) = \alpha(h)\,(\tilde{e}_\alpha, \tilde{f}_\alpha) = \alpha(h)$$

Moreover, $\alpha^\vee = h_\alpha = \dfrac{2\tilde{h}_\alpha}{(\tilde{h}_\alpha, \tilde{h}_\alpha)} = \dfrac{2\tilde{h}_\alpha}{\alpha(\tilde{h}_\alpha)}$

So, when identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ via the non-deg. form $(\ ,\ )$, α corresponds to $\tilde{h}_\alpha$. Note that we can also transport $(\ ,\ )$ to a form on $\mathfrak{h}^*$, such that

$$(\alpha, \beta) = (\tilde{h}_\alpha, \tilde{h}_\beta).$$

<u>Prop</u>: Let $\alpha, \beta \in \Phi$

(1) $\langle \beta, \alpha^\vee \rangle = \dfrac{2(\beta, \alpha)}{(\alpha, \alpha)}$

(2) $(\alpha, \beta) \in \mathbb{Q}$

(3) $(\alpha, \alpha) > 0$

(4) $(\, , \,)_{\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}}$ is positive definite.

<u>Proof</u>: (1) We have

$$\langle \beta, \alpha^\vee \rangle = \beta(h_\alpha) = \beta\left(\frac{2\tilde{h}_\alpha}{(\tilde{h}_\alpha, \tilde{h}_\alpha)}\right) = \frac{2\beta(\tilde{h}_\alpha)}{(\tilde{h}_\alpha, \tilde{h}_\alpha)} = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

(2) Consider the $\mathfrak{s}_\alpha$-module $V = V_{\alpha,\beta} = \bigoplus \mathfrak{g}_{\beta + i\alpha}$.

Since $\tilde{h}_\alpha = [\tilde{e}_\alpha, \tilde{f}_\alpha]$, $\mathrm{tr}_V(\mathrm{ad}(\tilde{h}_\alpha)) = 0$. On the

other hand

(2)-(4) Consider for $h, h' \in \mathfrak{h}$

$$(h, h') = tr_{\mathfrak{g}}(\operatorname{ad} h \operatorname{ad} h') = \sum_{\beta \in \Phi} \beta(h)\beta(h')$$

Then, for $h = h' = \tilde{h}_\alpha$, we get

$$(h_\alpha, h_\alpha) = \sum_{\beta \in \Delta} \beta(\check{h}_\alpha)^2$$

Hence, we obtain

$$1 = (\tilde{h}_\alpha, \hat{h}_\alpha) \sum_{\beta \in \Delta} \left(\frac{\beta(\hat{h}_\alpha)}{(\tilde{h}_\alpha, \check{h}_\alpha)} \right)^2$$

$$= (\tilde{h}_\alpha, \tilde{h}_\alpha) \underbrace{\sum \left(\tfrac{1}{2} \langle \beta, \alpha^\vee \rangle \right)^2}_{\in \mathbb{Q}_{>0}}$$

So $(\tilde{h}_\alpha, \tilde{h}_\alpha) = (\alpha, \alpha) \in \mathbb{Q}_{>0}$. Hence

$$(\alpha, \beta) = \frac{1}{2} \langle \beta, \alpha^\vee \rangle (\alpha, \alpha) \in \mathbb{Q}.$$

Moreover, we see that h_α is a rational multiple of $\tilde{h}_\alpha$. Hence, for $\mathbb{Q} \subset R \subset \mathbb{C}$

$$\mathfrak{h}_R = \sum R \tilde{h}_\alpha = \sum R h_\alpha .$$

To see that $(\, , \,)_{\mathfrak{h}_\mathbb{R} \times \mathfrak{h}_\mathbb{R}}$ is pos. def., let $h \in \mathfrak{h}_\mathbb{R}$. Then, as above

$$(h, h) = \sum_{\beta \in \Phi} \underbrace{(\underbrace{\beta(h)}_{\in \mathbb{R}})^2}_{\mathbb{R}_{\geq 0}} \geq 0 .$$

Since $(\, ,)$ is non-deg., the statement follows $\square$

Remark: The Killing form on $\mathfrak{h}$ and $\mathfrak{h}^*$ can be reconstructed from the abstract root system, via

$$(\alpha^\vee, \beta^\vee) = \sum_{\gamma \in \Delta} \langle \gamma, \alpha^\vee \rangle \langle \gamma, \beta^\vee \rangle$$

Dually,

$$(\alpha, \beta) = \sum_{\gamma \in \Delta} \langle \alpha, \gamma^\vee \rangle \langle \beta, \gamma^\vee \rangle$$

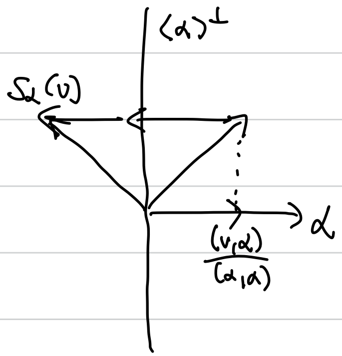
I.6. Euclidean root systems

<u>Recall</u>: let $(V, (,))$ be an Euclidean v.s.

let $0 \neq \alpha \in V$ Then the reflection through the orthogonal

hyperplane of is given by

$$S_\alpha : v \longmapsto v - 2 \frac{(v, \alpha)}{(\alpha, \alpha)}$$



Def: A Euclidean root system is a tuple

$$(V, (,), \Phi)$$

(1) $(V, (,))$ is a Euclidean v.s. generated by Φ, and $0 \notin \Phi$

(2) $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z} \quad \forall \alpha, \beta \in \Phi$

(3) s_α permutes $\Phi \quad \forall \alpha \in \Phi$

(4) $\mathbb{R}\alpha \cap \Phi = \{\pm \alpha\} \quad \forall \alpha \in \Phi$

Rem: One can translate between abstract and Euclidean root systems:

$$(V, (\,,\,), \Phi) \longleftrightarrow (V, \Phi, \Phi^\vee, (\;)^\vee)$$

"$\longrightarrow$" Define $\alpha^\vee = 2\frac{(\alpha, -)}{(\alpha, \alpha)}$

"$\longleftarrow$" Define $(v, v') = \sum_{\alpha \in \Phi^\vee} \langle v, \alpha^\vee \rangle \langle v', \alpha^\vee \rangle$

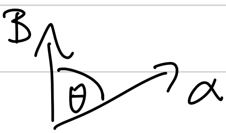
We will not prove this.

For a root system that comes from a semisimple Lie algebra

$$(\mathfrak{h}^*_{\mathbb{R}}, \underset{\substack{\uparrow \\ \text{Killing form}}}{(\,,\,)}, \Phi) \Longleftrightarrow (\mathfrak{h}^*_{\mathbb{R}}, \Phi, \Phi^\vee, (\;)^\vee)$$

Recall: let $(V, (\,,\,))$ be an Euclidean v.s. and $\alpha, \beta \in V$. Then

$$\frac{(\beta,\alpha)}{(\alpha,\alpha)} = \frac{\|\beta\|}{\|\alpha\|} \cos\theta$$



In particular, if α, β are roots in a root system

$$\mathbb{Z} \ni \langle \beta, \alpha^\vee \rangle \langle \alpha, \beta^\vee \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \frac{2(\alpha,\beta)}{(\beta,\beta)} = 4\cos^2\theta$$

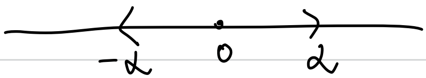
The only possibilities for this are (assuming $\|\beta\| \geq \|\alpha\|$)

$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$	$\lvert s_\alpha s_\beta \rvert$
0	0	$\pi/2$	undet	2
1	1	$\pi/3$	1	3
-1	-1	$2\pi/3$	1	3
1	2	$\pi/4$	2	4
-1	-2	$3\pi/4$	2	4
1	3	$\pi/6$	3	6
-1	-3	$5\pi/6$	3	6

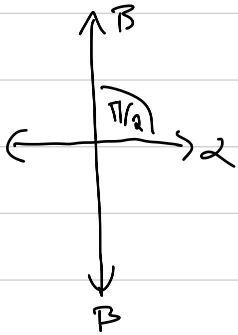
Exercise: Look up the definitions of irreducible root system, isomorphism of root system, direct sum, rank, root string

Lecture 12

Example (1) A_1

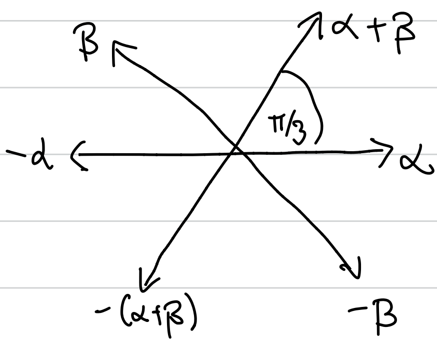


(2) $A_1 \times A_1$



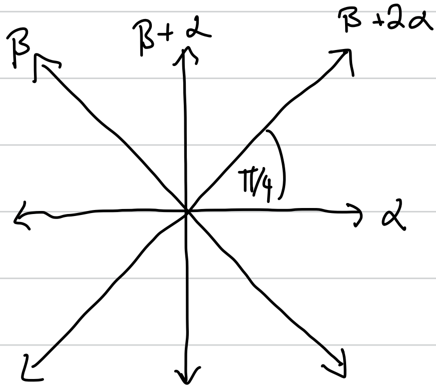
$\|\alpha\| \overset{?}{=} \|\beta\|$

(3) A_2 root system



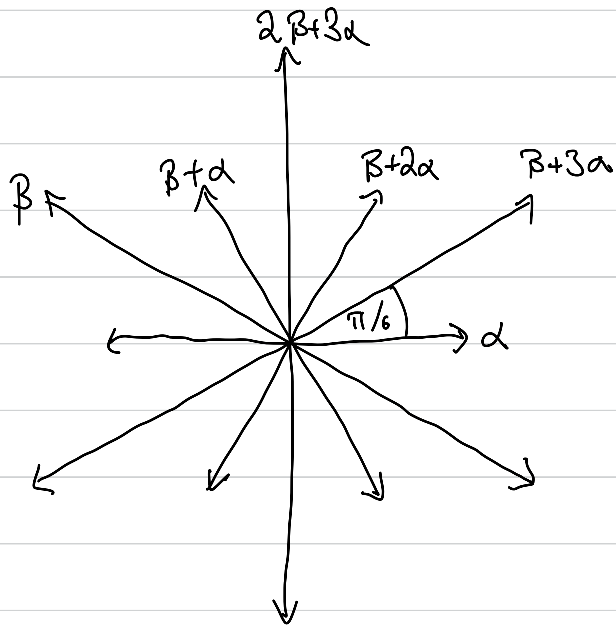
$\|\alpha\| = \|\beta\|$

(4) $B_2 = C_2$



$\sqrt{2}\|\alpha\| = \|\beta\|$

(5) G_2



$\sqrt{3}\|\alpha\| = \|\beta\|$

1.7 Very brief introduction to finite reflection groups

Let $(V, (\,,\,))$ be a f.d. Euclidean v.s.

<u>Recall</u>: (1) $g \in GL(V)$ is called orthogonal if

$(g-, g-) = (-,-)$ and $O(V) = \{ g \in GL(V) \mid g \text{ orthogonal} \}$.

(2) For each hyperplane $H \subset V$ there is a unique

reflection $s_H \in O(H)$ with fixed points $V^{s_H} = H$.

Vice versa, a reflection $s \in O(H)$ yields a hyperplane

$V^s \subset V$, so that we obtain a bijection between reflections

and hyperplanes

Def A: (1) A finite reflection group $W \subset O(V)$ is a finite subgroup generated by reflections.

(2) The hyperplane arrangement $\mathcal{H}$ associated to W is the set of all hyperplanes $H = V^s$ for $s \in W$ a reflection.

(2) A Weyl chamber or alcove is a connected component of

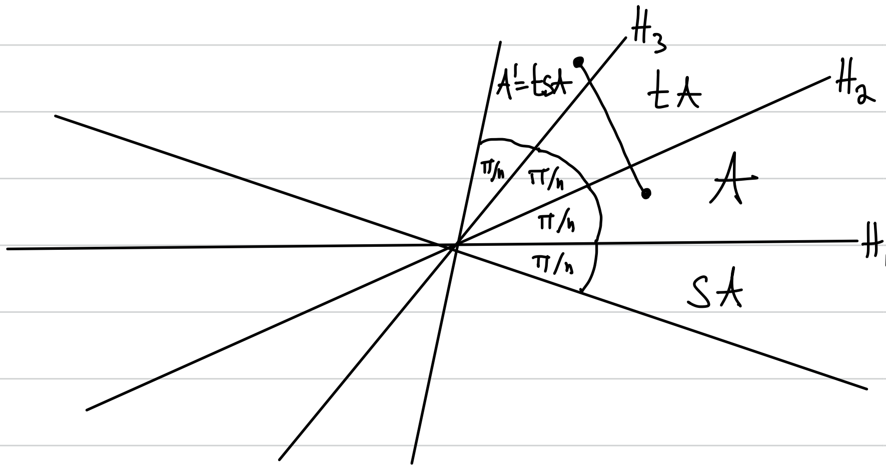
$$V - \bigcup_{H \in \mathcal{H}} H$$

(3) For an alcove $A \subset V$, a hyperplane $H \in \mathcal{H}$ called a wall of A if $\langle \bar{A} \cap H \rangle_{\mathbb{R}} = H$ and

we denote the set of walls by $\mathcal{H}_A$. Moreover we denote $S(A) = \{ s_H \mid H \in \mathcal{H}_A \} \subset W$ and call it the set of <u>simple reflections</u> associated to A.

(4) For two alcoves A, A' we denote by $d(A, A')$ the number of hyperplanes in $\mathcal{H}$ intersecting a line segment $[a, a']$ for generic points $a \in A$, $a' \in A'$

Example: (1) The dihedral group $D_{2n} \subset O(\mathbb{R}^2)$



Fix an alcove A, then $\mathcal{H}_A = \{H_1, H_2\}$,

$S(A) = \{s = s_{H_1}, t = s_{H_2}\}$. Then st is a clockwise rotation by $2\pi/n$. D_{2n} is generated by s, t and admits a presentation

$$W = \langle s, t \mid (st)^n = 1 \rangle$$

There is a bijection

$D_{2n} \to \{\text{alcoves}\}$, $\quad w \mapsto wA$

and $|D_{2n}| = 2n$.

Exercise: Express all $2n$ elements in terms of D_{2n}.

(2) Let $V = \mathbb{R}^n$ and denote $H_{ij} = \{(x_1, \ldots, x_n) \mid x_i = x_j\}$

for $i \neq j$. Then $s_{H_{ij}} = s_{ij} = (i\,j)$ permutes the

coordinates i and j. The associated finite reflection

group is S_n, the symmetric group on n elements.

$A = \{(x_1, \ldots, x_n) \mid x_1 < x_2 < \cdots < x_n\}$ is an alcove.

Exercise: Describe all alcoves

(3) Let $(V, (\,,\,), \Phi)$ be an Euclidean root system, then $W = \{ s_\alpha \mid \alpha \in \Phi \}$ is a finite reflection group, called the <u>Weyl group</u>.

Exercise: Determine $W, \mathcal{H}, \dots$ for the rank 2 root systems in 1.6. Example.

Thm: Let $W \subset O(V)$ be a finite reflection group and $A \subset V$ be an alcove and let $S = S(A) \subset W$.

(1) S generates W

(2) Let $w \in W$ and write $w = s_1 \cdots s_r$ such that $s_i \in S$ and r is minimal. Then $r = d(A, wA)$.

(3) W acts free and transitively on the set of alcoves, so

$$W \to \{\text{alcoves}\}, \quad w \mapsto wA$$

is a bijection

(4) Each reflection $t \in W$ is conjugate to a simple reflection $s \in S$

Proof Omitted $\square$

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<u>Def</u> B: For a fixed alcove $A \subset V$ and $S = S(A)$ we call $\ell(w) = \ell_A(w) = \ell_S(w) = d(A, wA)$ the <u>length</u> of w and an expression $w = s_1 \cdots s_r$ with $r = \ell(w)$ and $s_i \in S$ a <u>reduced expression</u> $\square$

<u>Remark</u>: For a finite reflection group, (W, S) forms a <u>Coxeter system</u>, that is, W admits a presentation of the form

$$W = \langle s \in S \mid s^2 = 1, (st)^{m_{s,t}} = 1 \rangle$$

for $m_{s,t} \geqslant 2$.

Coxeter systems are also characterized by the following technical but extremely useful property.

<u>Lemma</u> A (Exchange lemma): Let $s_1, \dots, s_r \in S$ and $t \in W$ a reflection such that

$$\ell(ts_1 \cdots s_r) < \ell(s_1 \cdots s_r)$$

then there is an index i, s.t.,

$$ts_1 \cdots s_r = s_1 \cdots s_{i-1} s_{i+1} \cdots s_r \qquad \square$$

<u>Remark</u>: There is a unique longest element $w_0 \in W$ (so $\ell(w_0) \geq \ell(w) \; \forall w \in W$). Moreover $\ell(w_0) = |W|$

Def C: Fix an alcove A, $S = S(A)$.

(1) We denote by $\leq = \leq_A$ the smallest transitive order on the set of alcoves, such that

$A' \leq tA'$ for all reflections $t \in W$ with $d(A,A') < d(A,tA')$

(2) We denote by $\leq = \leq_A$ the smallest transitive order on W, such that

$w \leq tw$ for all reflections $t \in W$, with $\ell(w) \leq \ell(sw)$ $\square$

Lemma B: (1) Let $w = s_1 \cdots s_{\ell(w)}$ a reduced expression.

Then $\{x \in W \mid x \leq w\} = \{s_{i_1} \cdots s_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq \ell(w)\}$

(2) For $x, y \in W$ and $s \in S$ then 0, 3 or 4 of the following inequalities hold

$$x \leq y \qquad x \leq sy$$

$$sx \leq y \qquad sx \leq sy \qquad \square$$

<u>Exercise</u>: Study the Bruhat order for the examples we discussed so far.

1.8 Bases of root systems

<u>Def</u> A: let $\Phi \subset V$ be a root system, then

(1) $W = \langle s_\alpha \rangle \subset GL(V)$ is called the Weyl group.

(2) A subset $\Delta \subset \Phi$ is called a basis if

(1) Δ is a basis of V

(2) If $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \Phi$ then either $n_\alpha \in \mathbb{Z}_{\geq 0}$ $\forall \alpha$

or $n_\alpha \in \mathbb{Z}_{\leq 0}$ $\forall \alpha$

(3) A subset $\Phi^+ \subset \Phi$ is called a <u>system of positive roots</u> if

(1) $\Phi = \Phi^+ \uplus \overbrace{-\Phi^+}^{=\Phi^-}$

(2) If $\beta = \sum_{\alpha \in \Phi^+} n_\alpha \alpha \in \Phi$ for $n_\alpha \in \mathbb{Z}_{\geq 0}$, then $\beta \in \Phi^+$.

<u>Def B</u>: Given an abstract root system $(V, \Phi, \Phi^\vee, (\ )^\vee)$ we denote by $(V^*, \Phi^\vee, \Phi, (\ )^\vee)$ the (Langlands) <u>dual root system</u> $\square$

<u>Rem</u>: For a Euclidean root system $((V, (\ ,\ )), \Phi)$ the dual root system is $((V, (\ ,\ )), \Phi^\vee)$ with

$$\Phi^\vee = \{ \alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \mid \alpha \in \Phi \} \qquad \square$$

In particular, we obtain (a priori) two Weyl groups $W = \langle s_\alpha \rangle \subset GL(V)$, $W^\vee = \langle s_{\alpha^\vee} \rangle \subset GL(V^*)$.

However $W \to W^\vee$, $s_\alpha \mapsto s_{\alpha^\vee}$ is an isomorphism.

lecture 13

Thm: Given a root system $\Phi \subset V$, there is a commutative diagram of bijections

$$
\begin{tikzcd}
\{\text{alcoves in } V\} \arrow[r, shift left, "7"] \arrow[d, shift left, "3"] & \{\text{set of simple reflections } S \subset W\} \arrow[l, shift left, "8"] \arrow[dl, shift left, "6"] \\
\{\text{bases of } \Phi\} \arrow[u, shift left, "4"] \arrow[ur, shift left, "5"] \arrow[d, shift left, "1"] & \\
\{\text{system of positive roots in } \Phi\} \arrow[u, shift left, "2"] &
\end{tikzcd}
$$

1: $\Delta \mapsto \mathbb{Z}_{\geq 0}\Delta \cap \Phi$

2: $\Phi^+ \mapsto \{\alpha \in \Phi^+ \mid \alpha \notin \mathbb{Z}_{\geq}(\Phi^+ \setminus \{\alpha\})\}$

3: $A \subset V \mapsto \Delta(A) = \{\alpha \in \Phi \mid \langle \alpha, A \rangle > 0 \ \forall\, a \in A$ and H_α is a wall of $A\}$

$5: \Delta \mapsto S = \{s_\alpha \mid \alpha \in \Delta\}$

$7: A \mapsto S(A)$

Proof: Omitted $\square$

Exercise: (1) What is 4, 6, 8?

(2) Study this in the example A_2 $\square$

Cor: (1) The maps in the Theorem are compatible with the action of W. In particular, W acts on all the sets freely and transitively.

(2) Each root $\alpha \in \Phi$ is contained in at least one basis

<u>Proof</u>: Use Thm and Thm A I.6.

We will also make use of the following useful definition.

<u>Def</u> C: Let $\Delta \subseteq \Phi$ be a basis of a root system.

For $\lambda, \lambda' \in V$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda \in \mathbb{Z}_{\geq 0} \Delta$. $\square$

<u>Rem.</u> With this definition $\alpha \in \Phi$ is in $\Phi^+ \iff \alpha > 0$ $\square$

Lemma Let $\Delta \subset \Phi$ be a set of simple roots. Let $\alpha, \beta \in \Delta$ be simple roots such that $\|\alpha\| \leq \|\beta\|$. Then $\langle \alpha, \beta^\vee \rangle$, $\langle \beta, \alpha^\vee \rangle$ can take the following values

$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	θ	$\|\beta\|^2/\|\alpha\|^2$	$\lvert s_\alpha s_\beta \rvert$
0	0	$\pi/2$	undet	2
-1	-1	$2\pi/3$	1	3
-1	-2	$3\pi/4$	2	4
-1	-3	$5\pi/6$	3	6

Proof: By the corresponding Table in I.6, if $\langle \alpha, \beta^\vee \rangle > 0$, then either $\langle \alpha, \beta^\vee \rangle$ or $\langle \beta, \alpha^\vee \rangle = 1$. WLog $\langle \alpha, \beta^\vee \rangle = 1$. Then $s_\beta(\alpha) = \alpha - \langle \alpha, \beta^\vee \rangle \beta = \alpha - \beta \in \Phi$.

But this is a contradiction to the definition of a basis. $\square$

1.9 Cartan matrix, Dynkin diagram and Classification

Def: An $n\times n$-matrix $A=(a_{ij})_{i,j=1,\dots,n}$ is called a Cartan matrix if

(C1) $a_{ii}=2$ for $1\le i\le n$

(C2) $a_{ij}\in\mathbb{Z}_{<0}$ for $i\neq j$

(C3) $a_{ij}=0 \Rightarrow a_{ji}=0$

(C4) $A=DB$ for D diagonal, B positive definite

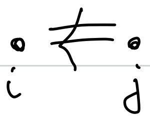
The Dynkin diagram associated to a Cartan matrix

decorated graph with:

(D1) The vertex set is $1, \dots, n$

(D2) Connect $i \neq j$ with $\max\{|a_{ij}|, |a_{ji}|\}$ lines

(D3) If $i \neq j$ and $|a_{ij}| \geq 2$, add an arrow



If the Dynkin diagram is a connected, we refer to it as

(C5) A is indecomposable $\square$

<u>Example</u>: (1) Let $(V, (\,,\,), \Phi)$ be a root system. Let $\Delta = \{\alpha_1, \dots, \alpha_n\}$ be a choice of positive roots. Let $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$. Then $A = (a_{ij})_{1 \leq i,j \leq n}$ is a Cartan matrix: (C1)–(C3) are straightforward

To see $C(4)$, let $B = ((\alpha_i, \alpha_j))_{1 \le i,j \le n}$

$D = \mathrm{diag}\left(\frac{2}{(\alpha_i, \alpha_i)}\right)_{1 \le i \le n}$. Then $A = BD$.

Moreover $\underset{i}{\circ} \Rrightarrow \underset{j}{\circ}$ if $\|\alpha_i\| > \|\alpha_j\|$.

(2) Vice versa, one can construct a root system from a Cartan matrix (Exercise: How?) and there are bijection

$$\begin{array}{c}
\{\text{root systems with chosen basis}\}/\text{iso} \\
\updownarrow \\
\{\text{Cartan matrixs}\}/\text{reordering of index set} \\
\updownarrow \\
\{\text{Dynkin diagram}\}/\text{iso}
\end{array}$$

Thm: The Dynkin diagrams of indecomposable Cartan matrices are of the form

Classical

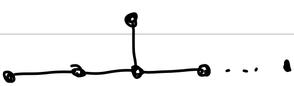
 A_n $\quad \mathfrak{sl}_{n+1}$

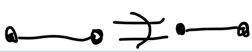
 B_n $\quad \mathfrak{so}_{2n+1}$

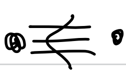
 C_n $\quad \mathfrak{sp}_{2n}$

 D_n $\quad \mathfrak{so}_{2n}$

Exceptional

 E_6, E_7, E_8

 F_4

 G_2

Lecture 14

Proof: We first classify the diagrams disregarding the length of roots, which amounts to classifying subsets $\mathcal{U} = \{\varepsilon_1, \dots, \varepsilon_n\} \subset V$ of unit vectors in an Euclidean vector space such that

(1) $\{\varepsilon_1, \dots, \varepsilon_n\}$ is linearly independent

(2) $(\varepsilon_i, \varepsilon_j) \leq 0$ for $i \neq j$

(3) $4(\varepsilon_i, \varepsilon_j)^2 \in \{0, 1, 2, 3\}$ for $i \neq j$.

We call such a subset admissible. We associate a graph Γ to an admissible subset with vertices $i = 1, \dots, n$ and $4(\varepsilon_i, \varepsilon_j)^2$ edges between i and j

for $i \neq j$. For example, $\{\frac{\alpha_i}{\|\alpha_i\|} \mid i=1,\dots,n\}$ is an admissible subset if $\Delta = \{\alpha_1,\dots,\alpha_n\}$ a set of simple roots in a root system. The graph Γ is the simply the Dynkin diagram minus the decorations "<". We call Γ the Coxeter diagram.

We now collect some properties of admissible sets and their Coxeter diagrams.

(1) Each subset of an admissible set is admissible.

Proof is clear

(2) Let N be the number of pairs of vertices connected by at least one edge. Then $N < n$:

To see this, consider $\varepsilon = \sum_{i=1}^{n} \varepsilon_i$. Then

$0 \leq (\varepsilon, \varepsilon) = n + 2 \sum_{i<j} (\varepsilon_i, \varepsilon_j) \leq n - N$

since $i \doteq j$ implies that $2(\varepsilon_i, \varepsilon_j) \leq -1$.

(3) Γ contains no cycle:

A cycle yields an admissible set violating (2)

(4) No more than 3 edges originate from one vertex:

Let $\varepsilon \in \mathfrak{A}$ be connected to vertices $\eta_1, \dots, \eta_k$ by at least one edge

So $(\varepsilon, \eta_i) < 0$. By (3) no pair η_i, η_j can be connected, so $(\eta_i, \eta_j) = 0$ for $i \neq j$.

Now choose a unit vector $\eta_i \in \langle \varepsilon, \eta_1, \dots, \eta_k \rangle \cap \langle \eta_1, \dots, \eta_k \rangle^{\perp}$. Then $\varepsilon = \sum_{i=0}^{k} (\varepsilon, \eta_i)\, \eta_i$ and $(\varepsilon, \eta_0) \neq 0$. Moreover,

$$1 = (\varepsilon, \varepsilon) = \sum_{i=0}^{k} (\varepsilon, \eta_i)^2 > \sum_{i=1}^{k} (\varepsilon, \eta_i).$$

Hence $\#$ edges connected to $\varepsilon = \sum_{i=1}^{k} 4 (\varepsilon, \eta_i) < 4$

(5) The only connected subgraph of Γ containing a triple edge is of the form  :

Follows from (4)

(6) Let $\{\varepsilon_1, \dots, \varepsilon_k\} \subset A$ with subgraph a simple chain

$$\circ - \circ - \circ \; \cdots \; \circ - \circ$$

Then $A' = A - \{\varepsilon_1, \dots \varepsilon_k\} \cup \{\varepsilon\}$ with $\varepsilon = \sum_{i=1}^{k} \varepsilon_k$

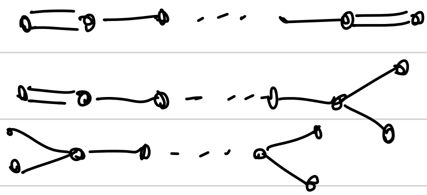
is also admissable. We can hence contract simple chains to a point:

A' is clearly still linearly independent. Since $2(\varepsilon_i, \varepsilon_{i+1}) = -1$, we get

$$(\varepsilon, \varepsilon) = k + 2 \sum_{i<j} (\varepsilon_i, \varepsilon_j) = k - k + 1 = 1$$

Now $\eta \in A - \{\varepsilon_1, \dots, \varepsilon_k\}$ is connected to at most one ε_i. Then $(\eta, \varepsilon) = (\eta, \varepsilon_i)$ or 0.

(7) Γ contains no subgraph of the form

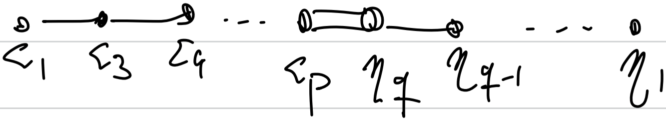


;

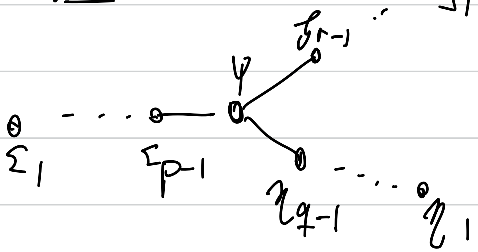
Follows from (6) and (4).

(8) Each connected component of Γ has the form

(a) 

(b) 

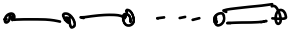
(c) 

(d) 

:

Follows from (7) and (3)

(9) The only connected component of type (b) in (8) must be of the form  (F_4) or

 (B_n, C_n):

Let $\varepsilon = \sum_{i=1}^{p} i\varepsilon_i$, $\eta = \sum_{i=1}^{q} i\eta_i$. Then

$$(\varepsilon,\varepsilon) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p(p+1)/2$$

$(\eta,\eta) = q(q+1)/2$ and $(\varepsilon,\eta)^2 = p^2q^2/2$

By the Schwartz inequality

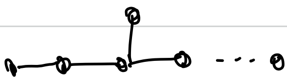
$$p^2q^2/2 = (\varepsilon,\eta)^2 < (\varepsilon,\varepsilon)(\eta,\eta) = p(p+1)q(q+1)/4$$

which implies $(p-1)(q-1) < 2$. So either $p=q=2$

or $p=1, q=0$ or $p=0, q=1$

7.10) The only connected components of type (d) are

 (D_n) and

 $(E_n,\ n=6,7,8)$:

Let $\varepsilon = \sum i\varepsilon_i$, $\eta = \sum i\eta_i$, $\zeta = \sum i \zeta_i$.

As in the proof of (4), we obtain

$$1 = (\varphi,\varphi) > \frac{(\varphi,\varepsilon)^2}{(\varepsilon,\varepsilon)} + \frac{(\varphi,\eta)^2}{(\eta,\eta)} + \frac{(\varphi,\zeta)^2}{(\zeta,\zeta)}$$

$$> \frac{1}{2}\left(1-\frac{1}{p}+1-\frac{1}{q}+1-\frac{1}{r}\right)$$

which is equivalent to

$$1/p + 1/q + \frac{1}{r} > 1.$$

Assume $1/p \leq 1/q \leq 1/r$. We may also assume $1/r \leq \frac{1}{2}$, since else we are in type A_n.

Then the only solutions (p,q,r) are of the form

$(p,2,2)$, $(3,3,2)$, $(4,3,2)$, $(5,3,2)$. $\Box$

Exercise: Look up: Wilhelm Killing

Die Zusammensetzung der stetigen endlichen Transformationsgruppen. Zweiter Theil

and A.J. Coleman

The Greatest Mathematical Papers of All Time

Lecture 15

1.10. An extended example: $\mathfrak{sl}_n$

Consider the Lie algebra

$$\mathfrak{g} = \mathfrak{sl}_3 = \{X \in \mathfrak{gl}_3 \mid \operatorname{tr} X = 0\}$$

which has dimension $3 \times 3 - 1 = 8$. We choose the Cartan

$$\mathfrak{h} = \mathfrak{sl}_3 \cap \left\{ \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix} \right\}$$

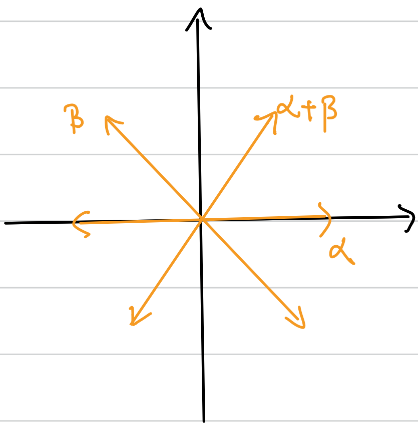
Denote by $\varepsilon_i : \left\{ \begin{bmatrix} x_1 & & \\ & x_2 & \\ & & x_3 \end{bmatrix} \right\} \to \mathbb{C} : \varepsilon_i^*$ the obvious maps for $i = 1, \dots, 3$.

We then obtain the roots and coroots:

γ	$\mathfrak{g}_\gamma$	H_γ	$\gamma^\vee$
$\alpha = \varepsilon_1 - \varepsilon_2$	$\begin{vmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{vmatrix}$	$\begin{vmatrix} 1 & & \\ & -1 & \\ & & 0 \end{vmatrix}$	$\alpha^\vee = \varepsilon_1^* - \varepsilon_2^*$
$\beta = \varepsilon_2 - \varepsilon_3$	$\begin{vmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{vmatrix}$	$\begin{vmatrix} 0 & & \\ & 1 & \\ & & -1 \end{vmatrix}$	$\beta^\vee = \varepsilon_1^* - \varepsilon^*$
$\alpha+\beta = \varepsilon_1 - \varepsilon_3$	$\begin{vmatrix} 0 & & 1 \\ & 0 & \\ & & 0 \end{vmatrix}$	$\begin{vmatrix} 1 & & \\ & 0 & \\ & & -1 \end{vmatrix}$	$(\alpha+\beta)^\vee = \varepsilon_1^* - \varepsilon_3^*$

and $\mathfrak{g}_{-\gamma} = (\mathfrak{g}_\gamma)^{tr}$.

The root system has the form (visualised as Euclidian space):



and $\Pi = \{\alpha, \beta\}$ yields a choice of simple roots. The associated Cartan matrix and Dynkin diagram are:

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

The α-string through β is of the form

$$\beta,\ \beta+\alpha = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

Correspondingly, we obtain an action of

$$\mathfrak{s}_\alpha = \left\langle e_\alpha = \begin{vmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{vmatrix},\ h_\alpha = \begin{vmatrix} 1 & & \\ & -1 & \\ & & 0 \end{vmatrix},\ f_\alpha = e_\alpha^{tr} \right\rangle_{\mathbb{C}}$$

$$= \left\{ \left[\begin{array}{cc|c} \multicolumn{2}{c|}{A} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0 & 0 & 0 \end{array}\right] \;\middle|\; A \in \mathfrak{sl}_2 \right\} \cong \mathfrak{sl}_2$$

on the space

$$V_{\alpha,\beta} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\alpha} = \left\{ \begin{bmatrix} 0 & 0 & \boxed{v} \\ & 0 & \\ & & 0 \end{bmatrix} \;\middle|\; v \in \mathbb{C}^2 \right\}$$

The action is given by:

$$\left[\begin{bmatrix} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{bmatrix} , \begin{bmatrix} 0 & \boxed{v} \\ & 0 \end{bmatrix} \right] = \begin{bmatrix} 0 & \boxed{Av} \\ & 0 \end{bmatrix}$$

This is exactly the sl_2-rep. $\mathbb{C} = L(-\langle \beta, \alpha^\vee \rangle) = L(1)$.

Now consider $\mathfrak{g} = \mathfrak{sl}_n$, $\mathfrak{h}$ = diagonal matrices

$\mathfrak{h}^* = \bigoplus \mathbb{C}\varepsilon_i / \sum \mathbb{C}\varepsilon_i$. See 1.1. Example.

Then $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ and

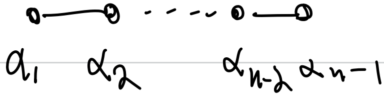
$\Phi^\vee = \{(\varepsilon_i - \varepsilon_j)^\vee = \varepsilon_i^* - \varepsilon_j^* = E_{ii} - E_{jj}\}$

Now choose $\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\}$.

Then $\langle \alpha_i, \alpha_j^\vee \rangle = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \end{cases}$ and the

Cartan matrix and Dynkin diagram are

$$\begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}$$



A_{n-1}

<u>Exercise</u>: What is Φ^+ and W?

1.11. An extended example: Sp_{2n}

Consider the symplectic matrix

$$J_4 = \begin{bmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{bmatrix}$$

and the associated symplectic Lie algebra

$$\mathfrak{g} = \mathfrak{sp}_4 = \{ X \in \mathfrak{gl}_4 \mid X J_4 + J_4 X^{tr} = 0 \}$$

$$= \left\{ \begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix} \;\middle|\; \begin{array}{l} A, B, C \in \mathbb{C}^{2\times 2} \\ B = B^{tr},\ C = C^{tr} \end{array} \right\}$$

which has dimension $4 + 2 \cdot 3 = 10$. We choose the Cartan

$$\mathfrak{h} = \left\{ \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & -x_1 & \\ & & & -x_2 \end{bmatrix} \right\}$$

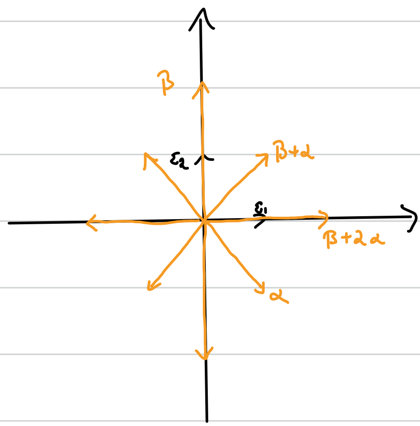
We denote the obvious bases of $\mathfrak{h}$ and $\mathfrak{h}^*$ by $\{\varepsilon_1^*, \varepsilon_2^*\}$ and $\{\varepsilon_1, \varepsilon_2\}$.

We then obtain the roots and coroots:

γ	$\mathfrak{g}_\gamma$	h_γ	$\gamma^\vee$
$\alpha = \varepsilon_1 - \varepsilon_2$	$\left(\begin{array}{cc\|cc} & a & & \\ & & & \\ \hline & & & \\ & & -a & \end{array}\right)$	$\left(\begin{array}{cc\|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right)$	$\alpha^\vee = \varepsilon_1 - \varepsilon_2$
$\beta = 2\varepsilon_2$	$\left(\begin{array}{cc\|cc} & & & \\ & & & a \\ \hline & & & \\ & & & \end{array}\right)$	$\left(\begin{array}{cc\|cc} & & & \\ & 1 & & \\ \hline & & & \\ & & & -1 \end{array}\right)$	$\beta^\vee = \varepsilon_2$
$\alpha+\beta = \varepsilon_1 + \varepsilon_2$	$\left(\begin{array}{cc\|cc} & & & a \\ & & a & \\ \hline & & & \\ & & & \end{array}\right)$	$\left(\begin{array}{cc\|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right)$	$(\alpha+\beta)^\vee = \varepsilon_1 + \varepsilon_2$
$2\alpha+\beta = 2\varepsilon_1$	$\left(\begin{array}{cc\|cc} & & a & \\ & & & \\ \hline & & & \\ & & & \end{array}\right)$	$\left(\begin{array}{cc\|cc} 1 & & & \\ & & & \\ \hline & & -1 & \\ & & & \end{array}\right)$	$(2\alpha+\beta)^\vee = \varepsilon_1$

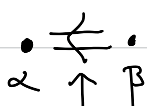
and $\mathfrak{g}_{-\gamma} = (\mathfrak{g}_\gamma)^{tr}$.

The root system has the form (visualised as Euclidian space):



and $\Delta = \{\alpha, \beta\}$ yields a choice of simple roots. The associated Cartan matrix and Dynkin diagram are:

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$



$(\alpha,\alpha) < (\beta,\beta)$

$A = DB$ for $D = \begin{pmatrix} 1/2 & \\ & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$

The α-string through β is of the form

$$\beta,\ \beta+\alpha,\ \underset{-\langle \beta, \alpha^\vee\rangle}{\underset{\parallel}{\beta+2\alpha}}$$

Correspondingly, we obtain an action of

$$\mathfrak{s}_\alpha = \left\langle e_\alpha = \left[\begin{array}{cc|cc} & 1 & & \\ & & & \\ \hline & & & \\ & & -1 & \end{array}\right],\ h_\alpha = \left[\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & \end{array}\right],\ f_\alpha = e_\alpha^{tr} \right\rangle_{\mathbb{C}}$$

$$= \left\{ \begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix} \;\middle|\; A \in \mathfrak{sl}_2 \right\} \cong \mathfrak{sl}_2$$

on the space

$$V_{\alpha,\beta} = \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\alpha} \oplus \mathfrak{g}_{\beta+2\alpha} = \left\{ \begin{bmatrix} & B \\ & \end{bmatrix} \;\middle|\; B = B^{tr} \right\}$$

The action is given by:

$$\left[\begin{bmatrix} A & \\ & -A^{tr} \end{bmatrix}, \begin{bmatrix} & B \\ & \end{bmatrix}\right] = \begin{bmatrix} & AB+BA^{tr} \\ & \end{bmatrix}.$$

This is exactly the sl_2-rep. $S^2(V) = L(-\langle\beta,\alpha^\vee\rangle) = L(2)$.

Similarly $\langle \alpha+\beta, \alpha^\vee\rangle = 0$ corresponds to the fact that $[s_\alpha, s_{\alpha+\beta}] = 0$

lecture 16

For $\mathfrak{g} = \mathfrak{sp}_{4n} = \{X \in \mathfrak{gl}_{2n} \mid X J_{2n} + J_{2n} X^{tr} = 0\}$

$$= \left\{ \begin{bmatrix} A & B \\ C & -A^{tr} \end{bmatrix} \;\middle|\; \begin{array}{l} A, B, C \in \mathbb{C}^{n \times n} \\ B = B^{tr},\ C = C^{tr} \end{array} \right\}$$

with $J_{2n} = \begin{vmatrix} & I_n \\ -I_n & \end{vmatrix}$. Then

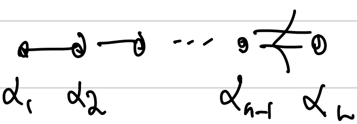
$\mathfrak{h} = \{\mathrm{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n)\}$ is a Cartan subalgebra

$\Phi = \{\pm \varepsilon_i \pm \varepsilon_j \mid 1 \le i, j \le n\} \setminus \{0\}$, we can choose

$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\} \cup \{\alpha_n = 2\varepsilon_n\}$. Then we

obtain the Cartan matrix and Dynkin diagram

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & -1 & & \\ & & -1 & 2 & -1 \\ & & & -2 & 2 \end{pmatrix}$$



C_n

Exercise: Determine root spaces, $\Phi^\vee$, Φ^+, W.

1.12 Orthogonal Lie algebras

For symmetric bilinear forms it is convenient to work with the following symmetric matrices

$$B_{2n} = \begin{pmatrix} & I_n \\ I_n & \end{pmatrix} \quad \text{and} \quad B_{2n+1} = \begin{pmatrix} B_{2n} & \\ & 1 \end{pmatrix}$$

Case 2n: We consider the orthogonal Lie algebra

$$so_{2n} = \{ X \in gl_{2n} \mid X B_{2n} + B_{2n} X^{tr} = 0 \}$$

$$= \left\{ \begin{vmatrix} A & B \\ C & -A^{tr} \end{vmatrix} \;\middle|\; \begin{matrix} B = -B^{tr}, \\ C = -C^{tr} \end{matrix} \right\}$$

with Cartan $h = \{ \text{diag}(x_1, \ldots, x_n, -x_1, \ldots, -x_n) \}$

Then $\underline{\Phi} = \{ \pm \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \}$

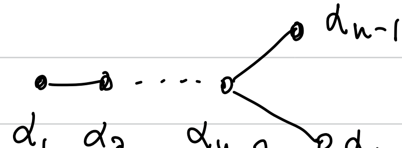
and we can choose

$$\Delta = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i = 1, \dots, n-1\} \cup \{\alpha_n = \varepsilon_{n-1} + \varepsilon_n\}$$

so that we obtain Dynkin diagram and Cartan matrix

$$\begin{pmatrix} 2 & -1 & & & & \\ -1 & \ddots & \ddots & & & \\ & \ddots & \ddots & -1 & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & \\ & & & -1 & & 2 \end{pmatrix}$$

D_n :



Case $2n+1$: In this case, we obtain

$$\mathfrak{so}_{2n+1} = \{X \in \mathfrak{gl}_{2n} \mid X B_{2n+1} + B_{2n+1} X^{tr} = 0\}$$

$$= \left\{ \begin{pmatrix} A & B & u \\ C & -A^{tr} & v \\ -v^{tr} & -u^{tr} & 0 \end{pmatrix} \;\middle|\; \begin{array}{l} B = -B^{tr}, \\ C = -C^{tr} \end{array} \right\}$$

$$\mathfrak{h} = \{\operatorname{diag}(x_1, \dots, x_n, -x_1, \dots, -x_n, 0)\}$$

$$\Phi = \{ \pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \} \cup \{ \pm \varepsilon_i \mid 1 \leq i \leq n \}$$

and we choose

$$\Delta = \{ \alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \leq i \leq n-1 \} \cup \{ \alpha_n = \varepsilon_n \}$$

so that

$$C = \begin{pmatrix} 2 & -1 & & & \\ -1 & \ddots & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -2 \\ & & & -1 & 2 \end{pmatrix} \qquad B_\ell : \underset{\alpha_1}{\circ} - \cdots - \underset{\alpha_{n-1}}{\circ} \Rightarrow \underset{\alpha_n}{\circ}$$

<u>Exercise:</u> Compute all the rest!

1.12 The classification Thm.

<u>Thm</u> (Killing classification) There are natural bijections

$$\begin{array}{c}\{\text{indecomposable Dynkin diagrams}\}/\text{iso} \\ 1 \downarrow \uparrow 2 \\ \{\text{indecomposable root systems}\}/\text{iso} \\ 3 \downarrow \uparrow 4 \\ \{\text{simple Lie algebras}\}/\text{iso}\end{array}$$

In particular, each simple Lie algebra is isomorphic to exactly one of the following Lie algebras

$A_n \quad sl(n+1), \quad n \geq 1$
$B_n \quad so(2n+1), \quad n \geq 2$
$C_n \quad sp(2n), \quad n \geq 3$
$D_n \quad so(2n), \quad n \geq 4$

or the exceptional Lie algebras e_6, e_7, e_8, f_4, g_2.

<u>Proof Idea</u>: We have seen how to construct maps 2 and 4. Moreover, we described root systems and Lie algebras for all classical Dynkin diagrams. One may also explicitly construct root systems / Lie algebras for the 5 exceptional cases. This shows the surjectivity of 2, 4. The rest of the proof relies on the following remark, which we will not prove.

<u>Remark</u>: (1) Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and simple roots $\Delta \subset \mathfrak{h}^*$. Then as a Lie algebra, $\mathfrak{g}$ has the following presentation:

<u>Generators</u> $\quad e_\alpha, h_\alpha, f_\alpha$ for $\alpha \in \Delta$

<u>Relations</u> for all $\alpha, \beta \in \Delta$:

(R1) $\quad [e_\alpha, f_\beta] = \delta_{\alpha,\beta} h_\alpha$

(R2) $\quad [h_\alpha, h_\beta] = 0$

(R3) $\quad [h_\alpha, e_\beta] = \langle \beta, \alpha^\vee \rangle e_\beta$

(R4) $\quad [h_\alpha, f_\beta] = -\langle \beta, \alpha^\vee \rangle e_\beta$

(R5) $\quad (\operatorname{ad} e_\alpha)^{1-\langle \beta, \alpha^\vee \rangle} e_\beta = 0 = (\operatorname{ad} f_\alpha)^{1-\langle \beta, \alpha^\vee \rangle} f_\beta$ B

(2) There are exceptional isomorphisms

$$so(3) \cong sp(2) = sl(2)$$

$$sp(4) \cong so(5)$$

$so(2) \cong \mathbb{C}$ is Abelian

$so(4) \cong sl(2) \times sl(2)$ is not simple and

$$so(6) \cong sl(4)$$

<u>Exercise</u>: Research the exceptional iso's!

Lecture 17

III Highest weight modules

III.1 Weight lattice and dot-action.

Let $\mathfrak{g} \supset \mathfrak{h}$ be a s.s. Lie algebra with fixed Cartan subalgebra.

Moreover let $\Delta \subset \Phi^+ \subset \Phi$ a fixed choice of positive and simple roots.

Def A (1) The <u>weight lattice</u> is the subset

$$X = \{ \lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \quad \forall \alpha \in \Phi^+ \}$$

(2) The set of <u>dominant weights</u> is

$$X^+ = \{ \lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \quad \forall \alpha \in \Phi^+ \}$$

(3) The <u>fundamental weight</u> $\omega_\alpha \in \mathfrak{h}^*$ associated to

$\alpha \in \Delta$ is defined by $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta} \quad \forall \beta \in \Delta$.

(3) The <u>root lattice</u> is $\mathbb{Z}\Phi \subset \mathfrak{h}^*$.

(4) For $\lambda, \mu \in \mathfrak{h}^*$ we write

$$\lambda \leq \mu \quad \iff \quad \mu - \lambda \in \mathbb{Z}_{\geq 0}\Phi^+ = \mathbb{Z}\Delta$$

<u>Remark</u>: (1) $\mathbb{Z}\Phi \subset X$ and

$$\mathbb{Z}\Phi = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha, \quad X = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\varpi_\alpha, \quad X^+ = \bigoplus_{\alpha \in \Delta} \mathbb{Z}_{\geq 0}\varpi_\alpha.$$

(2) The Weyl group acts on $\mathbb{Z}\Phi$ and X. The $\mathbb{R}_{\geq 0}$-span of X^+ forms the alcove associated to the choice $\Delta \subset \Phi$.

Def B: We denote the half-sum of positive roots by

$$\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$$

and define the <u>dot-action</u> of W on $\mathfrak{h}^*$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho .$$

Prop. (1) For $\alpha \in \Delta$, we have $s_\alpha(\rho) = \rho - \alpha$ and $\langle \rho, \alpha^\vee \rangle = 1$

(2) $\rho = \sum_{\alpha \in \Delta} \omega_\alpha \in X^+$.

(3) The dot-action preserves $\mathbb{Z}\Phi$ and X

Proof (1) We have $s_\alpha(\Phi^+) = \Phi^+ \setminus \{\alpha\} \cup \{-\alpha\}$.

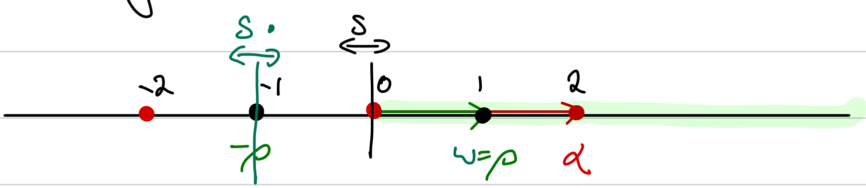
so that $s_\alpha(\rho) = \rho - \alpha$. Moreover

$$-\alpha = s_\alpha(\rho) - \rho = \rho - \langle \rho, \alpha^\vee \rangle \alpha - \rho = -\langle \rho, \alpha^\vee \rangle \alpha.$$

Hence $-\langle \rho, \alpha^\vee \rangle = 1$.

(2) + (3) follow from (1) $\square$

<u>Example</u>: (1) $\mathfrak{g} = \mathfrak{sl}_2$:



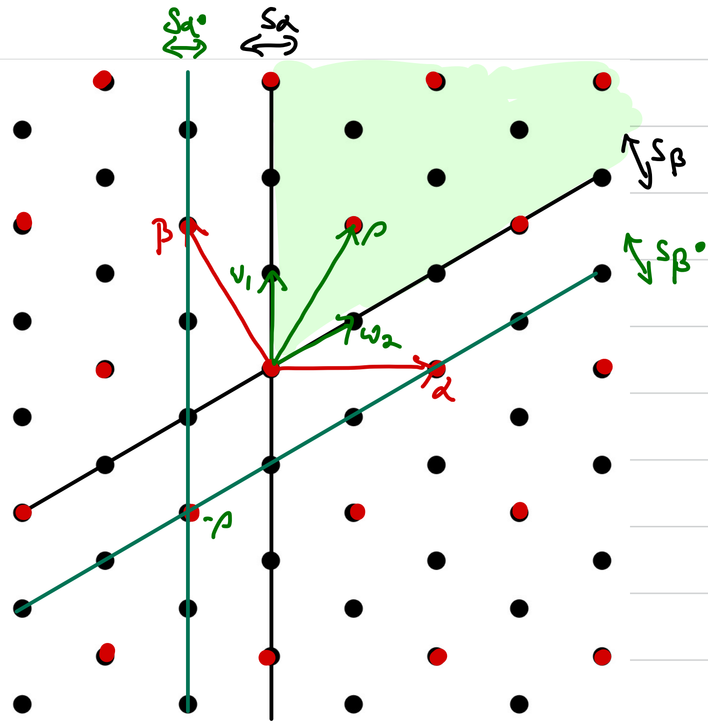
If we identify $\mathbb{C} = \mathfrak{h}^*$, then $X = \mathbb{Z}$ and $\mathbb{Z}\Phi = 2\mathbb{Z}$.
$1 \mapsto \omega$

and $X^+ = \mathbb{Z}_{\geq 0}$. Moreover $s(\lambda) = -\lambda$ and

$s \cdot \lambda = -\lambda - 2$. In particular for $\lambda \in X^+$ we get

$$M(s \cdot \lambda) \hookrightarrow M(\lambda) \to L(\lambda), \text{ see } 0.4.$$

(2) $\quad \mathfrak{g} = \mathfrak{sl}_3$:



II.2 Borel subalgebras

Let $\mathfrak{g}$ be a s.s. Lie algebra.

Def A: A Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a maximal solvable subalgebra.

Prop: Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgebra. Then $\mathfrak{h} = \{X \in \mathfrak{b} \mid X \text{ is semi-simple}\}$ is a Cartan subalgebra.

Proof: Exercise.

We now fix a Cartan $\mathfrak{h} \subset \mathfrak{g}$.

Lecture 18

Thm: The following are bijections:

$$\{\text{system of positive roots}\} \underset{1}{\downarrow} \overset{2}{\uparrow} \{\text{Borel subalgebras } \mathfrak{b} \supset \mathfrak{h}\}$$

1: $\Phi^+ \mapsto \mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$

2: $\mathfrak{b} \mapsto \Phi^+ = \{\alpha \mid \mathfrak{g}_\alpha \subset \mathfrak{b}\}$

Proof: We show that 1 is well-defined:

For $\gamma = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \Phi^+$, denote $ht(\gamma) = \sum n_\alpha$.

Let $\mathfrak{n}_{\geq i} = \bigoplus_{ht\gamma \geq i} \mathfrak{g}_\gamma$. Then $\mathfrak{n}_{\geq i} \subset \mathfrak{b}$ is an ideal. Moreover $\mathfrak{n}_{\geq i}/\mathfrak{n}_{\geq i+1}$ is Abelian. This shows that $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is solvable.

now let $\mathfrak{g} \supset \mathfrak{p} \supsetneq \mathfrak{b}$. Then since $\mathfrak{p}$ is $\mathfrak{h}$-stable, there is a $\mathfrak{g}_\alpha \subset \mathfrak{p}$ with $\alpha \leq 0$. But then $\mathfrak{sl}_2 \cong \mathfrak{s}_\alpha \subset \mathfrak{p}$ and $\mathfrak{p}$ is not solvable $\square$

<u>Example</u>: In the examples of type ABCD we considered in 1.9–1.11, the Borel subalgebra corresponds to the subsets of upper-triangular matrices in the respective Lie algebra.

<u>Remark</u>: The set $\mathcal{B} = \{\mathfrak{b} \mid \mathfrak{b} \subset \mathfrak{g} \text{ Borel subalgebra}\}$ is a smooth projective algebraic variety called the <u>flag variety</u> (of $\mathfrak{g}$). We will discuss this next term!

Prop: For a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$,

$\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ is nilpotent. $\square$

Remark: We will often denote $\mathfrak{b}^-$ and $\mathfrak{n}^-$ for the algebras corresponding to $\Phi^- = -\Phi^+$. In particular

$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. (for $\mathfrak{n}^+ = \mathfrak{n}$) $\square$

II.3 (Highest) weights

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a s.s. Lie algebra with fixed Borel and Cartan. Denote $\Phi \supset \Phi^+ \supset \Delta$ the set of (positive) and simple roots. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$.

Def A: Let M be a $\mathfrak{g}$-rep.

(1) M is called a weight module if

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$

(2) We denote by $P(M) = \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$ the set of weights of M.

(3) We call $\lambda \in P(M)$ a highest weight of M

if λ is maximal in $P(V)$ with respect to $\leq$.

(4) A vector $0 \neq v \in M_\lambda$ is called a highest weight vector if $\mathfrak{b} v \subset M_\lambda$ or (equivalently) if $\mathfrak{n} v = 0$

(5) A weight module is called a <u>highest weight</u> module (of weight λ) if it is generated by a highest weight vector $0 \neq v \in M_\lambda$. $\square$

<u>Thm</u>: Let L be a f.d. rep.

(1) The set of weights $P(L)$ is invariant under the action of the Weyl group W.

(2) $P(L) \subset X$

(3) If L is irreducible, then L is a highest weight module for some dominant weight $\lambda \in X^+$.

<u>Proof</u>: Let $\lambda \in P(L)$ and $\alpha \in \Delta$. Denote by $S_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle_{\mathbb{C}}$ the copy of $\mathfrak{sl}_2$ corresponding to α. Moreover let $0 \neq v \in L_\lambda$. Since L is finite dimensional our discussion in 0.4 implies that $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ and $0 \neq f_\alpha^{\langle \lambda, \alpha \rangle} v \in L_{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha}$

Hence, $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \in P(V)$. Since $W = \langle s_\alpha \mid \alpha \in \Delta \rangle$, we obtain (1). Also (2) holds, since $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\forall \alpha \in \Delta$.

For (3), note that L is a weight module (since it is f.d.). Choose a highest weight $\lambda \in \mathcal{P}(V)$ and vector $v \in L_\lambda$. Then $\mathfrak{g}_\alpha v = 0$ for $\alpha \in \Phi^+$ since $\lambda + \alpha \geq \lambda$. Hence $\mathfrak{n} v = 0$. Moreover v generates L, since L is irreducible. To see that $\lambda \in X^+$, we use 0.4. again. Namely, $\mathfrak{g}_\alpha v = 0$ implies $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$. $\square$

II.4 Verma modules

Let $\lambda \in \mathfrak{h}^*$. Then we obtain the 1-dim. $\mathfrak{h}$-rep $\mathbb{C}_\lambda$.

By the natural map $\mathfrak{b} \to \mathfrak{h} \cong \mathfrak{b}/\mathfrak{n}$, we

can extend this to a $\mathfrak{b}$-rep. (so $\mathfrak{n}\,\mathbb{C}_\lambda = 0$).

Def: Let $\lambda \in \mathfrak{h}^*$. Then the Verma module of

highest weight λ is defined as

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \qquad \square$$

Prop: (1) $M(\lambda)$ is a highest weight module of

weight λ and $v_+ = 1 \otimes 1$ is a highest weight vector.

(2) The map

$$\mathcal{U}(\mathfrak{n}^-) \to M(\lambda), \quad u \mapsto u v_+$$

is an isomorphism of $\mathcal{U}(\mathfrak{b}^-)$-modules.

(3) $M(\lambda)$ is a weight module and

$$P(M(\lambda)) = \lambda - \mathbb{Z}_{\geq 0} \Phi_+$$

(4) If λ is a highest weight in a module M,

then the map

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), M_\lambda) \to M_\lambda, \quad f \mapsto f(v^+)$$

is a bijection.

Proof: (1) is clear.

(2) Recall from II.2 Remark, that

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{n}^- \oplus \mathfrak{b}.$$

Hence, by the PBW theorem (0.2. Thm),

$$\mathcal{U}(\mathfrak{g}) \cong \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})$$

which is even true as $(\mathcal{U}(\mathfrak{n}^-), \mathcal{U}(\mathfrak{b}))$ bimodule. Hence

$$\begin{aligned} M(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda &= \mathcal{U}(\mathfrak{n}^-) \otimes \mathcal{U}(\mathfrak{b}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda \\ &= \mathcal{U}(\mathfrak{n}^-) \otimes \mathbb{C}_\lambda \\ &\xleftarrow{\sim} \mathcal{U}(\mathfrak{n}^-) \end{aligned}$$

(3) Let $\{\alpha_1, \dots, \alpha_k\} = \Phi^+$ be any ordering of the positive roots. Let $f_i = f_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ be a generator of the root space. Then by the PBW theorem, the set

$$f_1^{n_1} \cdots f_n^{n_k}, \quad n_i \in \mathbb{Z}_{\geq 0}$$

form a basis of $U(\mathfrak{n}^-)$. By (2), hence

$$f_1^{n_1} \cdots f_n^{n_k} v_+, \quad n_i \in \mathbb{Z}_{\geq 0}$$

forms a basis of of $M(\lambda)$, where the basis vector has weight

$$\lambda - \sum n_i \alpha_i \in \lambda - \mathbb{Z}\Phi^+.$$

(4) Exercise $\square$

Lecture 19

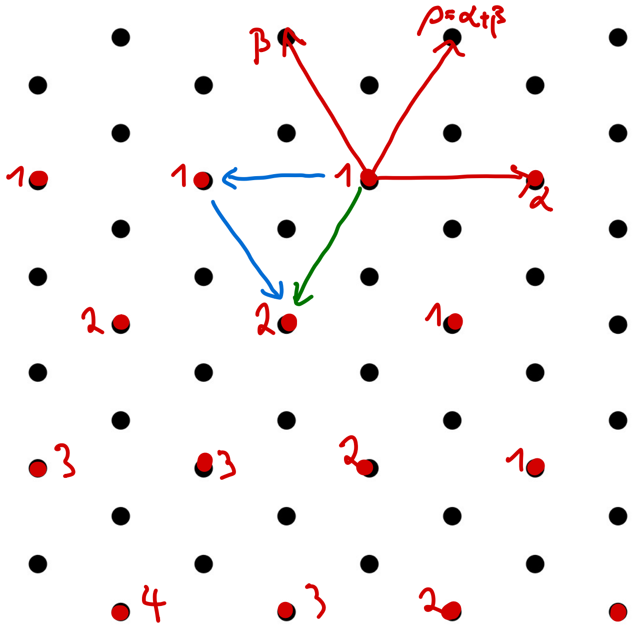
Remark: The dimension the weight space

$$\dim M(\lambda)_{\lambda-\mu} = \dim U(\mathfrak{n}^-)_{-\mu} = P(\mu)$$

$= \#$ ways μ can be expressed as sum of positive roots

is called the Kostant partition function.

Example: We consider $M(0)$ for $\mathfrak{g} = \mathfrak{sl}_3$:



Thm A: Let $\lambda \in \mathfrak{h}^*$.

(1) The Verma module $M(\lambda)$ contains a maximal proper submodule $\operatorname{rad} M(\lambda)$

(2) The map

$$\mathfrak{h}^* \longrightarrow \{\text{simple highest weight modules}\}/\cong$$

$$\lambda \longmapsto M(\lambda)/\operatorname{rad} M(\lambda)$$

is a bijection.

Proof: Let $N \subset M(\lambda)$ be a submodule. Then $N = \bigoplus_{\mu \leq \lambda} N_\mu$. If $N_\lambda \neq 0$, then $N = M$. Hence, for each proper submodule $N \subsetneq M$, $N \subset \bigoplus_{\mu < \lambda} M_\mu$.

Hence the sum of two proper submodules is proper again. Hence $\operatorname{rad} M(\lambda) = \sum_{N \subsetneq M} N$ is the maximal proper submodule.

(2) Exercise! $\square$

<u>Thm B</u>: Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Delta$ such that $s_\alpha \cdot \lambda \leq \lambda$.

Then there is an injection

$$M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda).$$

<u>Proof</u>: Assume $s_\alpha \cdot \lambda < \lambda$, (else $s_\alpha \cdot \lambda = \lambda$).

Since $s_\alpha \cdot \lambda = s_\alpha(\lambda) - \alpha = \lambda - (\langle \lambda, \alpha^\vee \rangle + 1)\alpha$,

$\lambda - s_\alpha \cdot \lambda = (\langle \lambda, \alpha^\vee \rangle + 1)\alpha$. Hence $n = \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$.

Again, let $s_\alpha = \langle e_\alpha, h_\alpha, f_\alpha \rangle \cong sl_2$.

$\mathcal{U}(s_\alpha) v^+ \cong M_{sl_2}(n)$ and by 0.4

(see also II.1. Example (1)), $f_\alpha^{n+1} v^+$ generates

the sl_2-submodule $M_{sl_2}(-n-2)$. In particular

$f_\alpha^{n+1} v \in M(\lambda)_{s_\alpha \cdot \lambda}$ and $e_\alpha f_\alpha^{n+1} v^+ = 0$.

Moreover $x_\beta f_\alpha^i v^+ = 0$ for all $i \in \mathbb{Z}_{>0}$ and

$\beta \in \Phi^+ \setminus \{\alpha\}$, since $i\alpha - \beta$ is not a

sum of positive roots and hence $M_{\lambda - i\alpha + \beta} = 0$.

In particular for $x_\beta f_\alpha^{n+1} v^+ = 0$. Hence

$f_\alpha^{n+1} v^+$ is a highest weight vector and we obtain our map $M(s_\alpha \cdot \lambda) \to M(\lambda)$.

To see that this map is injective one uses that $U(\mathfrak{n})$ has no zero divisors by the PBW-theorem and that $M(s_\alpha \cdot \lambda)$ and $M(\lambda)$ are free of rank 1 over $U(\mathfrak{n})$ $\square$

Lecture 20

II.4. Classification of f.d. simple modules

Def: Let $\mathfrak{a} \subset \mathfrak{g}$ be a subalgebra in a Lie-algebra and M a $\mathfrak{g}$-representation. Then $v \in M$ is called $\mathfrak{a}$-finite if the subspace $U(\mathfrak{a})v \subset M$ generated by v under the $\mathfrak{a}$-action is finite-dimensional.

Lemma: The set of $\mathfrak{a}$-finite vectors in a $\mathfrak{g}$-rep. M is stable under the action of $\mathfrak{g}$ and hence a subrepresentation.

Pf: Exercise.

Thm: The map

$$X^+ \longrightarrow \{\text{simple f.d. } \mathfrak{g}\text{-reps}\}/\cong$$

$$\lambda \longmapsto L(\lambda) = M(\lambda)/\operatorname{rad} M(\lambda)$$

is a bijection

Pf: By II.3. Thm (3), each simple f.d. rep. L is highest weight for some $\lambda \in X^+$. By II.4. Thm A, we obtain that $L = L(\lambda)$.

It remains to show that $L(\lambda)$ is f.d. for $\lambda \in X^+$!

For this we show that $W(P(L(\lambda))) = P(L(\lambda))$.

Since $P(L(\lambda)) \subset \lambda - \mathbb{Z}_{\geq 0}\Phi_+$, this implies that $P(L(\lambda))$ is finite and hence $L(\lambda)$ is f.d.

For this let $v^+ \in M(\lambda)$ be a highest weight vector and $\alpha \in \Delta$. Denote by $\overline{v}^+ \in L(\lambda) = M(\lambda)/\mathrm{Rad}\, M(\lambda)$ the image. Since $\lambda \in X^+$, $s_\alpha \cdot \lambda \leq \lambda$. Hence as in II.4. Thm B, $f_\alpha^{n+1} v^+ \in \mathrm{Rad}\, M(\lambda)$ and $f_\alpha^{n+1} \overline{v}^+ = 0$ for $n = \langle \lambda, \alpha^\vee \rangle$. Hence $U(\mathfrak{s}_\alpha) \overline{v}_+ \subset L(\lambda)$ is finite dimensional and non-trivial. By the previous lemma (using that $L(\lambda)$) is simple, $L(\lambda)$ is hence the sum of finite dimensional

S_α - representations. In particular $S_\alpha P(L(\lambda)) = P(L(\lambda))$.

(see 0.4 and II 3. Thm (1)). Hence $W P(L(\lambda)) = P(L(\lambda))$ $\square$

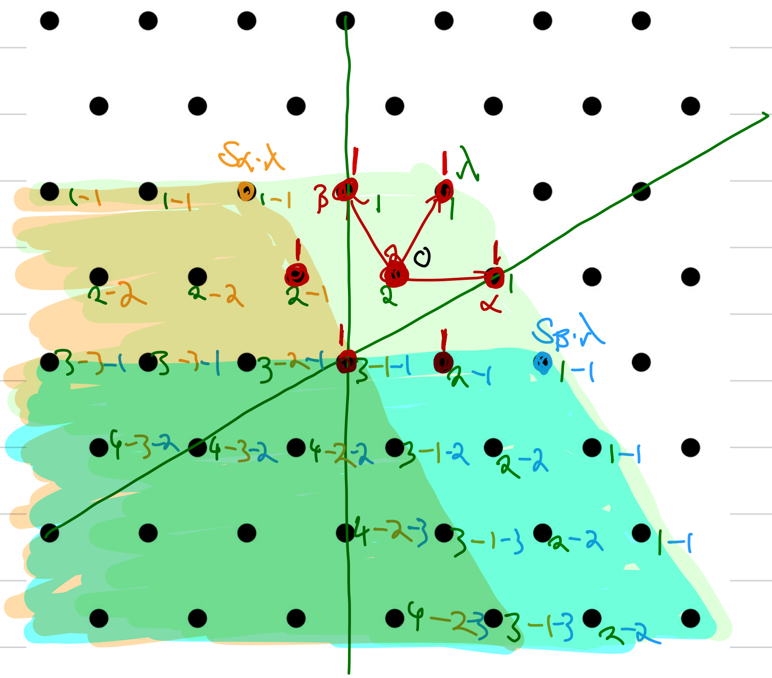
<u>Remark</u>: In fact, for $\lambda \in X^+$, $\mathrm{rad}(M(\lambda))$ is generated by the images of the maps $M(s_\alpha \cdot \lambda) \to M(\lambda)$.

We hence obtain an exact sequence

$$\bigoplus_{s \in S} M(s \cdot \lambda) \to M(\lambda) \to L(\lambda) \to 0$$

The sequence can be completed to the so-called BGG resolution of $L(\lambda)$.

Example: (1) We consider $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = \rho = \alpha + \beta$



Then $L(\rho) = (\mathfrak{sl}_3, ad) \cong$

Moreover, $L(\omega_\alpha = \varepsilon_1) = \mathbb{C}^3$, $L(\omega_\beta = \varepsilon_1 + \varepsilon_2) = (\mathbb{C}^3)^*$, ...

(2) For $\mathfrak{sl}_n$, $L(\omega_i) = \Lambda^i \mathbb{C}^n$, where

$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ is the fundamental weight

associated to $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, and $1 \leq i \leq n$.

11.5 Formal characters

<u>Def</u>: (1) Denote by $e^\lambda \in \mathrm{Fun}(\mathfrak{h}^*, \mathbb{Z})$ the indicator function of λ. Let $\mathcal{E} \subset \mathrm{Fun}(\mathfrak{h}^*, \mathbb{Z})$ the subspace of all functions that vanish outside some $\bigcup_{i=1}^{s}(\lambda_i - \mathbb{Z}_{\geq 0}\Phi^+)$. Then $\mathcal{E}$ is an algebra with respect to the convolution product

$$\sum_\lambda c_\lambda e^\lambda \cdot \sum_\lambda c'_\lambda e^\lambda = \sum_\lambda \Big(\sum_{\mu+\nu=\lambda} c_\mu c'_\nu\Big) e^\lambda$$

(2) Let M be a weight module. Then the (<u>formal</u>) <u>character</u> is

$$\mathrm{ch} M = \sum_{\lambda \in \mathfrak{h}^*} (\dim M_\lambda)\, e^\lambda$$

$\square$

<u>Proposition</u>: (1) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of weight modules, then $\mathrm{ch} M = \mathrm{ch} M' + \mathrm{ch} M''$

(2) For M, N weight modules with $\operatorname{ch} M, \operatorname{ch} N \in \mathcal{E}$,

we have $\operatorname{ch}(M \otimes N) = (\operatorname{ch} N)(\operatorname{ch} M) \in \mathcal{E}$

(3) For $\lambda \in \mathfrak{h}^*$, we have

$$\operatorname{ch} M(\lambda) = e^{\lambda} \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha} + e^{-2\alpha} + \dots)$$

$$= e^{\lambda} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})^{-1}$$

Proof (1) + (2) Exercise.

(3) Use that as $U(\mathfrak{n}^-)$-module $M(\lambda) \cong U(\mathfrak{n}^-)$ and the PBW theorem. See II.4 Remark $\square$

Example: Let $\mathfrak{g} = \mathfrak{sl}_2$; $\lambda = n\rho \in X_+$ for $n \in \mathbb{Z}_{\geq 0}$.

Then we have the s.e.s

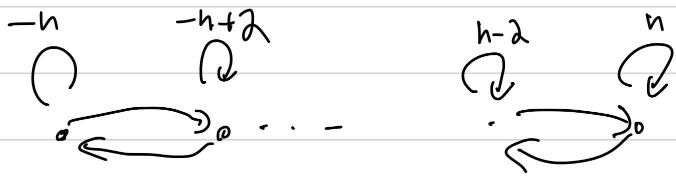
$$0 \to M(\underbrace{s\cdot\lambda}_{= -\lambda-2}) \to M(\lambda) \to L(\lambda) \to 0$$

Hence

$$\mathrm{ch}\, L(\lambda) = \mathrm{ch}\, M(\lambda) - \mathrm{ch}\,(M s\cdot\lambda)$$

$$= \frac{e^{\lambda} - e^{s\cdot\lambda}}{1 - e^{-\alpha}} = \frac{e^{(n+1)\rho} - e^{-(n+1)\rho}}{e^{\rho} - e^{-\rho}}$$

$$= e^{-n\rho} + e^{-n+2\rho} + \dots + e^{n\rho}$$



Lecture 21

The action of W on $\mathfrak{h}^*$ yields an action on formal characters, so $w(e^\lambda) = e^{w(\lambda)}$.

<u>Thm</u>: Let L be a f.d. $\mathfrak{g}$-rep. Then the character of L is W-invariant.

<u>Proof</u>: Let $\lambda \in P(L)$. It suffices to show that $L_\lambda \cong L_{s_\alpha \lambda}$ as vector spaces for $\alpha \in \Delta$.

Since L is f.d., $\lambda \in X$ by II.3 Thm (2), so $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$. Wlog, assume $\langle \lambda, \alpha^\vee \rangle \geq 0$ (else $\langle s_\alpha \lambda, \alpha^\vee \rangle \geq 0$). By $\mathfrak{sl}_2$-rep th. (see 0.4) there is an iso:

$$f_\alpha^{\langle \lambda, \alpha^\vee \rangle} : L_\lambda \to L_{\underbrace{\lambda - \langle \lambda, \alpha^\vee \rangle \alpha}_{= s_\alpha \alpha}} \qquad \square$$

II.6. Casimir action on Verma modules

We recall some facts on the Killing form and Casimir operator (see 0.8., 0.10, I.1, I.5, I.6). By definition the Killing form is

$$(x,y) = \mathrm{tr}(\mathrm{ad}\, x\ \mathrm{ad}\, y)$$

we saw that $(,)_{\mathfrak{h}\times\mathfrak{h}}$ is an inner product, which is W-invariant and transports to an inner product on $\mathfrak{h}^*$. Moreover $(,)$ pairs $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

By choosing $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ with $(x_\alpha, y_\alpha) = 1$ and an orthonormal basis h_i of $\mathfrak{h}$, we can write the Casimir operator as

$$C = \sum_{\alpha \in \Phi^+} y_\alpha x_\alpha + x_\alpha y_\alpha + \sum_i h_i^2$$

$$= \sum_{\alpha \in \Phi^+} 2 y_\alpha x_\alpha + [x_\alpha, y_\alpha] + \sum_i h_i^2 \qquad \in Z(U(\mathfrak{g}))$$

From this we may deduce the following generalization of of 0.11 Example (1).

<u>Thm</u>: The Casimir operator C acts on $M(\lambda)$ by multiplication with

$$c_\lambda = (\lambda+\rho, \lambda+\rho) - (\rho, \rho) = |\lambda+\rho|^2 + |\rho|^2$$

Proof: By II.4 Prop. (4), there is an isomorphism $\mathrm{End}_{\mathfrak{g}}(M(\lambda)) \xrightarrow{\sim} \mathrm{End}_{\mathbb{C}}(M(\lambda)_\lambda) = \mathbb{C}$. So we have to calculate how C acts on $0 \neq v_+ \in M(\lambda)_\lambda$.

Let $h \in \mathfrak{h}$, s.t. $\lambda = (h, -)$. Since $x_\alpha v_+ = 0$, we see that C acts on v_+ by

$$c_\lambda = \sum_{\alpha \in \Phi^+} \lambda([x_\alpha, y_\alpha]) + \sum \lambda(h_i)^2$$

$$= \sum_{\alpha \in \Phi^+} (h, [x_\alpha, y_\alpha]) + \sum (h, h_i)^2$$

$$= \left(\sum_{\alpha \in \Phi^+} ([h, x_\alpha], y_\alpha)\right) + \lambda(h)$$

$$= \left(\sum_{\alpha \in \Phi^+} \alpha(h)\right) + \lambda(h)$$

$$= 2\rho(h) + \lambda(h)$$

$$= (2\rho, \lambda) + (\lambda, \lambda)$$

$$= (\lambda+\rho, \lambda+\rho) - (\rho, \rho) \qquad \square$$

<u>Remark</u>. (1) C acts by c_λ on all subquotients of $M(\lambda)$, in particular also on $L(\lambda)$.

(2) For $w \in W$, we obtain

$$c_{w\cdot\lambda} = (w\cdot\lambda + \rho, w\cdot\lambda + \rho) - (\rho, \rho)$$

$$= (w(\lambda+\rho), w(\lambda+\rho)) - (\rho, \rho)$$

$$= (\lambda+\rho, \lambda+\rho) - (\rho, \rho) = c_\lambda . \qquad \square$$

117 Freudenthal formula

Using our computations with the Casimir operator, we obtain the following recursive character formula.

<u>Thm</u> (Freudenthal's formula). Let $\lambda \in X^+$. Then

$$(\dim L(\lambda)_\mu)(|\lambda+\rho|^2 - |\mu+\rho|^2) =$$

$$2 \sum_{\alpha \in \Phi^+} \sum_{j \geq 1} (\dim L(\lambda)_{\mu+j\alpha})(\mu+j\alpha, \alpha)$$

<u>Proof</u>: We first consider the case $\mathfrak{g} = \mathfrak{sl}_2 = \langle e, f, h \rangle_{\mathbb{C}}$.

Let $L = L(m\rho)$ be the $(m+1)$-dim. irreducible rep. Then by 0.4., fe acts on $L(m\rho)_{m\rho - i\alpha}$ by $(m-i+1)i$

which we write as

$$(m-i+1)i = \sum_{j\geq 1} (\dim L(m\rho)_{\mu+j\alpha})\langle \mu+j\alpha, \alpha^\vee\rangle$$

where $\mu = m\rho - i\alpha$. The RHS is zero if $L(m\rho)_\mu = 0$.

Using this one obtains that for each f.d. $\mathfrak{sl}_2$-rep,

$$(*) \quad \mathrm{tr}(fe|V_\mu) = \sum_{j\geq 1} (\dim V_{\mu+j\alpha})\langle \mu+j\alpha, \alpha^\vee\rangle.$$

Next we go back to the general case.

Recall that from $x_\alpha \in \mathfrak{g}_\alpha$, $y_\alpha \in \mathfrak{g}_{-\alpha}$ with $(x_\alpha, y_\alpha) = 1$

we obtained an $\mathfrak{sl}_2$-triple $e_\alpha = x_\alpha$, $h_\alpha = \alpha^\vee$, $f_\alpha = 2/(\alpha,\alpha)\, y_\alpha$, so

that

$$C = \underbrace{\sum_{\alpha\in\Phi^+} (\alpha,\alpha) f_\alpha e_\alpha}_{A} + \underbrace{\sum_{\alpha\in\Phi^+} [x_\alpha, y_\alpha] + \sum h_i^2}_{B}$$

As in the proof of 11.6.Thm, B acts on $L(\lambda)_\mu$ via $|\mu+\rho|^2-|\rho|^2$, so that

$$(**)\quad \mathrm{tr}(B|_{L(\lambda)_\mu}) = (\dim L(\lambda)_\mu)\,(|\mu+\rho|^2-|\rho|^2)$$

By $(*)$, the trace of the A-action is

$$(***)\quad \mathrm{tr}(A|_{L(\lambda)_\mu}) = \sum_{\alpha\in\Phi^+} (\alpha,\alpha) \sum_{j\geq 1} \dim L(\lambda)_{\mu+j\alpha} \langle \mu+j\alpha, \alpha^\vee\rangle$$

$$= 2\sum_{\alpha\in\Phi^+}\sum_{j\geq 1} \dim L(\lambda)_{\mu+j\alpha}\,(\mu+j\alpha,\alpha)$$

On the other hand, by 11.6.Thm,

$$(****)\quad \mathrm{tr}(C|_{L(\lambda)_\mu}) = (\dim L(\lambda)_\mu)\,(|\lambda+\rho|^2+|\rho|^2).$$

Putting together $(*)$ – $(****)$ yields the statement. ∎

Lecture 22

11.8 Kostant character formula

Lemma A: The Verma module $M(\lambda)$ has finite length and each simple subquotient is of the form $L(\mu)$ with $\mu \leq \lambda$ and $|\mu+\rho|^2 = |\lambda+\rho|^2$

Proof: The second statement follows from 11.6 and $P(M(\lambda)) = \lambda - \mathbb{Z}_{\geq 0}\Phi^+$. For the first statement, assume $\mu \leq \lambda$. Then $\mu = \lambda - \nu$ for $\nu \in \mathbb{Z}_{\geq 0}\Phi^+$. Now $|\mu+\rho|^2 = |\lambda+\rho|^2$ is equivalent to the equation

$$(\nu,\nu) + 2(\lambda+\rho,\nu) = 0.$$

Since $(\,,\,)$ is positive definite on $\mathbb{R}\Phi$, there are only finitely many solutions of this equation in $\mathbb{Z}\Phi$.

Since $P(M(\lambda))$ is a cone, each simple subquotient is a highest weight module $L(\mu)$ (Exercise: show this!!) and can appear at most $\dim M(\lambda)_\mu$ times in any composition series of $M(\lambda)$, so that we get

$$\text{length } M(\lambda) \leq \sum_{\substack{\mu \leq \lambda \\ |\mu+\rho|^2 = |\lambda+\rho|^2}} \dim M(\lambda)_\mu < \infty \qquad \square$$

Lemma B: Let $\mu \in X^+$ and $\nu \in X$. Then $|\nu| = |\mu|$ and $w\nu \leq \mu$ for all $w \in W$ implies $\nu \in W\mu$.

Proof: Since X^+ spans the fundamental alcove, $w\nu \in X^+$ for a unique $w \in W$. We hence assume $\nu \in X^+$ and just have to show that for $\mu, \nu \in X^+$, $\nu \leq \mu$ and $|\nu| = |\mu|$ implies $\nu = \mu$.

By assumption $\mu - \nu = \sum_{\alpha \in \Phi^+} c_\alpha \alpha$ with $c_\alpha \in \mathbb{Z}_{\geq 0}$.

Since $\mu, \nu \in X^+$, $(\mu, \alpha) \geq 0$ and hence $(\mu, \mu - \nu) \geq 0$. Hence

$$|\nu|^2 = |\mu|^2 + |\mu - \nu|^2 - 2(\mu, \mu - \nu) \leq |\mu|^2 + |\mu - \nu|^2$$

so that $|\nu|=|\mu|$ implies $|\nu-\mu|=0$ implies $\nu=\mu$. $\square$

<u>Lemma C</u>: Consider the <u>Weyl denominator</u>

$$R=\prod_{\alpha\in\Phi^+}(1-e^{-\alpha})=e^{-\rho}\prod_{\alpha\in\Phi^+}(e^{\alpha/2}-e^{-\alpha/2})=ch(M(0))$$

Let $w\in W$. Then

$$w(e^{\rho}R)=(-1)^{\ell(w)}e^{\rho}R$$

<u>Proof</u>: It suffices to treat the case $w=s_\alpha$ for $\alpha\in\Delta$

$$s_i(e^{\rho}\;R)=e^{\rho-\alpha}\prod_{\beta\in\Phi^+}(1-e^{-s_\alpha(\beta)})$$

$$=e^{\rho}e^{-\alpha}(1-e^{\alpha})R(1-e^{-\alpha})^{-1}$$

$$=-e^{\rho}R$$

where the second equation uses $s_\alpha(\Phi^+)=\Phi^+\setminus\{\alpha\}\cup\{-\alpha\}$ $\square$

so that $|\nu| = |\mu|$ implies $|\nu - \mu| = 0$ implies $\nu = \mu$. $\square$

<u>Thm A</u> (Kostant's character formula) Let $\lambda \in X^+$. Then

$$\operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch} M(w \cdot \lambda)$$

<u>Proof</u>: Let D be the (finite!) set of all $\nu \in X$ with $\mu \leq \lambda$ with $|\mu + \rho| = |\lambda + \rho|$. By Lemma A_1 for $\mu \in D$ we may write

$$\operatorname{ch} M(\mu) = \sum_{\substack{\nu \in D \\ \nu \leq \mu}} a_\nu^\mu \operatorname{ch} L(\nu)$$

for appropriate $a_\nu^\mu \in \mathbb{Z}_{\geq 0}$ and $a_\mu^\mu = 1$. Hence the a_ν^μ form a unipotent upper-triangular matrix

indexed by D which is partially ordered by $\leq$.
Hence, the matrix (a^{μ}_{ν}) has an inverse say (b^{μ}_{ν})
which is also unipotent upper-triangular. Hence

$$(*) \quad \operatorname{ch} L(\lambda) = \sum_{\mu \in D} b^{\mu}_{\lambda} \operatorname{ch} M(\mu) = R \sum_{\mu \in D} b^{\mu}_{\lambda} e^{\mu}$$

Since $\lambda \in X^+$, $L(\lambda)$ is finite dimensional and by
11.5. Thm $\operatorname{ch} L(\lambda)$ is W-invariant. We multiply
$(*)$ by $e^{\rho} R$ to obtain

$$(**) \quad e^{\rho} R \operatorname{ch} L(\lambda) = \sum_{\mu \in D} b^{\mu}_{\lambda} e^{\mu+\rho} = \sum_{\nu \in D+\rho} d_{\nu} e^{\nu}.$$

By lemma C, $w \in W$ acts by $(-1)^{\ell(w)}$ on the LHS.

Hence $d_\nu = (-1)^{\ell(w)} d_{w\nu}$, $d_{\lambda+\rho} = 1$. Moreover,

if $d_\nu \neq 0$, then $|\nu| = |\lambda+\rho|$ and $w\nu \leq \lambda+\rho$ for

all $w \in W$. By lemma B hence $\nu \in W(\lambda+\rho)$.

This implies $d_\nu \neq 0 \iff \nu = w(\lambda+\rho)$ and

$d_\nu = (-1)^{\ell(w)}$. This shows that $b_\lambda^\mu \neq 0 \iff \mu = w\cdot\lambda$

and $b_\lambda^\mu = (-1)^{\ell(w)}$ which we wanted to show. $\square$

Lecture 23

§1 Weyl's character formula and more Corollaries

By substituting the characters of Verma modules in Kostant's character formula, we obtain:

Thm A: (Weyl's character formula) Let $\lambda \in X^+$, then

$$ch(L(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})}$$

Cor. A: (Weyl's denominator formula)

$$\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}$$

Proof: Use Cor. A with $\lambda = 0$. Use $ch(L(0)) = e^0 = 1$ $\square$

<u>Cor B</u>: (Another form of Weyl's character formula) let $\lambda \in X^+$, then

$$\mathrm{ch}(L(\lambda)) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \cdot 0}} = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}}$$

Example: Let $\mathfrak{g} = \mathfrak{sl}_n$. Let $\varepsilon_i : \mathfrak{h} = \{(\diagdown)\} \to \mathbb{C}$

the projection to the i,i-entry. Then $\rho = \sum (n-i)\varepsilon_i$.

Let $\lambda \in X^+$. Since $\varepsilon_1 + \dots + \varepsilon_n = 0$, we can write

$\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_{n-1}\varepsilon_{n-1}$ with $\lambda_n = 0$. and

$\lambda \in X^+ \iff \lambda_1 \geq \dots \geq \lambda_{n-1}$.

We abbreviate $x_i = e^{\varepsilon_i}$ so that $e^\lambda = x_1^{\lambda_1} \cdots x_{n-1}^{\lambda_{n-1}} x_n^0$.

Recall that $W = S_n$ and $(-1)^{\ell(w)} = \mathrm{sgn}(w)$,

Moreover, $w(x_i) = x_{w(i)}$.

The Weyl character formula (in the form of Cor A),

then becomes

$$\operatorname{ch} L(\lambda) = \frac{\sum_{w \in S_n} \operatorname{sgn}(w)\, w(x_1^{\lambda_1+(n-1)} \cdots x_{n-1}^{\lambda_{n-1}+1} x_n^{\lambda_n})}{\sum_{w \in S_n} \operatorname{sgn}(w)\, w(x_1^{n-1} \cdots x_{n-1}^{1} \cdot x_n^{0})}$$

The parts of the fraction are determinants!

The numerator is given by

$$\det \begin{vmatrix} x_1^{\lambda_1+(n-1)} & \cdots & x_1^{\lambda_{n-1}+1} & x_1^{\lambda_n} \\ \vdots & & & \vdots \\ x_n^{\lambda_1+(n-1)} & \cdots & x_n^{\lambda_{n-1}+1} & x_n^{\lambda_n} \end{vmatrix}$$

and the numerator is the Vandermonde determinant

$$\det \begin{vmatrix} x_1^{n-1} & \cdots & x_1 & 1 \\ \vdots & & & \\ x_n^{n-1} & \cdots & x_n & 1 \end{vmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j) = e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})$$

which is a special case of the Weyl denominator formula (Cor. B). So we may write

$$\operatorname{ch} L(\lambda) = \frac{\det\left(x_i^{\lambda_j + (n-j)}\right)_{i,j}}{\det\left(x_i^{(n-j)}\right)_{i,j}}$$

The resulting polynomial is also known as the <u>Schur polynomial</u> S_λ.

This is a starting point to extremly rich combinatorics in Type A $\square$

Thm B: (Weyl's dimension formula). Let $\lambda \in X^+$, then

$$\dim L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda+\rho, \alpha^\vee \rangle}{\prod_{\alpha \in \Phi^+} \langle \rho, \alpha^\vee \rangle}$$

Proof: Denote by $\varepsilon : \mathbb{Z}[X] \to \mathbb{Z}$ the map $e^\lambda \mapsto 1$. Then $\dim L(\lambda) = \varepsilon \operatorname{ch} L(\lambda)$. It is tempting to apply ε to the numerator and denominator of the Weyl character formula, but this yields $\dim L(\lambda) = \frac{0}{0}$. Hence we use a form of l'Hospital's rule. For this, denote by $\partial_\alpha : \mathbb{Z}[X] \to \mathbb{Z}[X]$ the linear map given by $\partial_\alpha(e^\lambda) = \langle \mu, \alpha^\vee \rangle e^\lambda$. Then ∂_α is a derivation so $\partial_\alpha(fg) = \partial_\alpha(f)g + f\partial_\alpha(g)$ and all ∂_α

commute. Denote $D = \prod_{\alpha \in \Phi^+} \partial_\alpha$. Then

(1) $\varepsilon D e^\mu = \prod_{\alpha \in \Phi^+} \langle \mu, \alpha^\vee \rangle$

(2) $\varepsilon D e^{w\mu} = (-1)^{\ell(w)} \varepsilon D e^\mu \quad \forall w \in W.$

Now we apply εD to the equation

$$\left(e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})\right) \operatorname{ch} L(\lambda) = \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}$$

to obtain

$$\varepsilon D\left(e^\rho \prod (1 - e^{-\alpha})\right) \cdot \dim L(\lambda) = |W| \prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle.$$

Now we divide by the same equation for $\lambda = 0$ on

both sides and obtain the result. $\square$

Lecture 24 - skipped. Lecture 25

II.10 Tensor product decompositions

Recall that each f.d. module E decomposes as

$$E = \bigoplus_{\lambda \in X^+} L(\lambda)^{m_\lambda}$$

where $m_\lambda = \dim \mathrm{Hom}(L(\lambda), E) = \dim \mathrm{Hom}(M(\lambda), E)$. One denotes

$$[E : L(\lambda)] = m_\lambda$$

and calls this the multiplicity of $L(\lambda)$ in E.

<u>Lemma</u>: Assume that

$$\sum_{\lambda \in A} m_\lambda \operatorname{ch} M(\lambda) = \sum_{\lambda' \in B} n_\lambda \operatorname{ch} L(\lambda)$$

for some finite subsets $A, B \in \mathfrak{h}^*$. If $\lambda \in X^+$, then

$n_\lambda = m_\lambda$.

Proof: Interesting exercise! Show that for $\lambda \in X^+$

$\operatorname{ch} M(\lambda) \in \operatorname{ch} L(\lambda) + \langle \operatorname{ch} L(\mu) \mid \mu \in X - X^+ \rangle_{\mathbb{Z}}$ and

$\operatorname{ch} L(\lambda) \in \operatorname{ch} M(\lambda) + \langle \operatorname{ch} M(\mu) \mid \mu \in X - X^+ \rangle_{\mathbb{Z}}$ $\square$

Using this, we obtain the following practical formula for the important problem of tensor product decompositions:

Thm: (Klimyk's formula) Let $\lambda, \mu, \nu \in X^+$. Then

$$[L(\mu) \otimes L(\nu) : L(\lambda)] = \sum_{w \in W} (-1)^{\ell(w)} \dim L(\mu)_{\lambda - w \cdot \nu}$$

$$\leq \dim L(\mu)_{\lambda - \nu} .$$

Proof: Let E be a f.d. rep. of $\mathfrak{g}$ and denote by $P(E)$ the set of all weights. Then

$$\mathrm{ch}\, E \otimes L(\nu) = (\mathrm{ch}\, E)\, \mathrm{ch}(L(\nu))$$

$$= \sum_{\substack{w \in W \\ \tau \in P(E)}} (-1)^{\ell(w)} \dim E_\tau \, \mathrm{ch}\, M(w \cdot \nu + \tau)$$

where we use Kostant's character formula and

$$e^{\tau}\, \mathrm{ch}\, M(w \cdot \nu) = \mathrm{ch}\, M(w \cdot \nu + \tau).$$

Using the previous lemma and $\lambda \in X^+$, we obtain

$$[E \otimes L(\nu) : L(\lambda)] = \sum_{\substack{y \in W,\, \tau \in P(E) \\ y \cdot \nu + \tau = \lambda}} (-1)^{\ell(w)} \dim E_\tau$$

$$= \sum_{y \in W} (-1)^{\ell(w)} \dim E_{\lambda - y \cdot \nu}$$

With a similar argument, we get

$$[E \otimes \Delta(\nu), L(\lambda)] = \dim E_{\lambda-\nu}$$

which provides the inequality $\square$

<u>Remark</u>: If $\mu << \nu$, so for example

$$\langle \nu + z\mu, \alpha^\vee \rangle \geq 1 \quad \forall z \in W \text{ and } \alpha \in \Phi^+$$

then one might show that the inequality in Klimyk's formula is an equality and obtain the particularly nice formula

$$[L(\mu) \otimes L(\nu) : L(\lambda)] = \dim L(\mu)_{\lambda-\nu}$$

Cor (Steinberg's formula) Let $\mu, \nu, \lambda \in X^+$. Then

$$[L(\mu) \otimes L(\nu) : L(\lambda)] = \sum_{x,y \in W} (-1)^{\ell(xy)} \mathcal{P}(x \cdot \mu + y \cdot \nu - \lambda)$$

where $\mathcal{P}$ denotes Kostant's partition function.

Pf: Sketch: Use that by Kostant's character formula

$$\dim L(\mu)_\eta = \sum_{x \in W} (-1)^{\ell(x)} \mathcal{P}(x \cdot \mu - \eta)$$

and Klimyk's formula $\square$

Example: Consider $\mathfrak{g} = \mathfrak{sl}_3$ and let $\lambda = \omega_1 = \varepsilon_1$

$\nu = \omega_2 = -\varepsilon_3 = \varepsilon_1 + \varepsilon_2$. Then $L(\lambda) = \mathbb{C}^3$ and

$L(\nu) = L(\lambda)^* = (\mathbb{C}^3)^*$, moreover,

$$L(\lambda) \otimes L(\nu) = \mathbb{C}^3 \otimes (\mathbb{C}^3)^* = \mathrm{End}(\mathbb{C}^3) = \mathfrak{gl}_3$$

$$= \mathfrak{sl}_3 \oplus \mathbb{C}.$$

Exercise: confirm this using Klimyk's and Steinberg's formula.

II 11. Chevalley restriction and Harish-Chandra isomorphism

To understand the rep. theory of an algebra A, it is essential to understand it's center $A \supset Z = \{z \in A \mid za = az \ \forall a \in A\}$

Let M be an A-module, s.t. $\forall m \in M$ $Zv \subset M$ is finite dimensional. Then $M = \bigoplus_{\chi : Z \to \mathbb{C}} M_\chi$, where

$\chi : Z \to \mathbb{C}$ is an algebra homomorphism (usually called <u>central character</u>) and

$$M_\chi = \{m \in M \mid (z - \chi(z))^n m = 0 \text{ for } n \gg 0\}.$$

In other words, each $z \in Z$ acts on M_λ via Jordan blocks

$$\begin{pmatrix} \chi(z) & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \chi(z) \end{pmatrix}.$$

For $\mathfrak{g}$ semisimple, we already constructed the Casimir operator $C \in Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$. Our next goal is to compute all of Z.

We have

$$Z(\mathfrak{g}) = U(\mathfrak{g})^{\mathfrak{g}} = U(\mathfrak{g})^{G}$$

where $G = \mathrm{Int}(\mathfrak{g}) = \langle e^{\mathrm{ad}(x)} \mid x \in \mathfrak{g} \rangle \subset GL(\mathfrak{g})$.

By the PBW-theorem $U(\mathfrak{g})$ admits a filtration, such that $\mathrm{gr}\, U(\mathfrak{g}) = S(\mathfrak{g})$. Hence, we should first compute $S(\mathfrak{g})^{G}$.

For this we will write $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ and consider the projection map $\mathfrak{g} \to \mathfrak{h}$. This induces a map

res: $S(\mathfrak{g}) \to S(\mathfrak{h})$

<u>Thm</u> (Chevalley restriction theorem) The map res induces an isomorphism of algebras

$$S(\mathfrak{g})^G \longrightarrow S(\mathfrak{h})^W$$

<u>Proof</u>: We sketch the proof for $\mathfrak{g} = \mathfrak{gl}_n = \mathbb{C}^{n \times n}$; $G = GL_n$ and $W = S_n$. Here

$$S(\mathfrak{h})^W = \mathbb{C}[t_1, \dots, t_n]^{S_n} = \mathbb{C}[e_1, \dots, e_n]$$

where e_k is the kth symmetric polynomial

$$e_1 = t_1 + \cdots + t_n, \quad \ldots, \quad e_n = t_1 \cdots t_n.$$

On the other hand, we can consider

$$\det(\lambda I_n - X) = \lambda^n - c_1(X)\lambda^{n-1} + \cdots + (-1)^n c_n(X).$$

Since the characteristic polynomial is invariant under conjugation, we get $c_k(X) \in S(\mathfrak{gl}_n)^{GL_n}$. For example

$$c_1(X) = \operatorname{tr}(X), \quad \ldots, \quad c_n(X) = \det(X).$$

Note that for $X = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \in \mathfrak{h}$, we get

$$\det(\lambda I_n - X) = \prod_{i=1}^{n} (\lambda - t_i) = \lambda^n - e_1 \lambda^{n-1} + \cdots \pm e_n.$$

Hence $\operatorname{res}(c_k) = e_k$, so that res is surjective.

The rest is exercise! $\Box$

Next, consider the map

$$HC : Z(\mathfrak{g}) \subset U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+) \xrightarrow{pr} U(\mathfrak{h}) = S(\mathfrak{h})$$

Thm B (Harish-Chandra) HC is a homomorphism of algebras and induces an isomorphism

$$HC : Z(\mathfrak{g}) \to S(\mathfrak{h})^{W\bullet}$$

Pf: Let v_λ^+ be a highest weight vector of $M(\lambda)$

Write $z = HC(z) + u^- + u^+$, s.t. $u^- \in \mathfrak{n}^- U(\mathfrak{g})$, $u^+ \in U(\mathfrak{g})\mathfrak{n}^+$.

$$z \cdot v_\lambda^+ = HC(z) \cdot v^+ + \overbrace{u^- \cdot v^+}^{\text{weight} < \lambda} + \overbrace{u^+ \cdot v^+}^{=0}$$

$$= \lambda(HC(z))\, v_\lambda^+$$

so that we obtain

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, "\text{action}"] \arrow[dr, "HC"'] & \mathrm{End}(M(\lambda)) = \mathbb{C} \\ & S(\mathfrak{h}) \arrow[u, "\lambda"'] \end{tikzcd}$$

where the action map is a homomorphism of algebras. Since $S(\mathfrak{h}) \to \prod_{\lambda \in X} \mathrm{End}(M(\lambda))$ is injective, we see that HC is also a hom. of alg.

Now let $\lambda \in X$, $\alpha \in \Delta$ and assume wlog. that $s_\alpha \cdot \lambda \leq \lambda$.

Then by 11.4. Thm B, $M(s_\alpha \cdot \lambda) \hookrightarrow M(\lambda)$ and hence $Z(\mathfrak{g})$ acts on both by same scalar, so that

$$\lambda(HC(z)) = (s_\alpha \cdot \lambda)(HC(z))$$

so that $S_\alpha \cdot HC(z) = HC(z)$ and

$HC(Z(g)) \subset S(h)^{(W,\bullet)}$. We sketch why this

is actually an isomorphism:

Note that HC, on the associated

graded, exactly yields the Chevalley restriction map

$$S(g)^g \longrightarrow S(h)^W.$$

Now one may show that $\operatorname{gr} Z(g) = S(g)^g$ and

$\operatorname{gr} S(h)^{(W,\cdot)} = S(h)^W$. The theorem follows using that

HC is an iso iff its associated graded is $\square$